Competitive Prices in Large Markets with Private Information

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Abstract

Siga and Mihm (2020) characterize the information environments where prices can aggregate information in a competitive auction market with an atomless population of traders. In this paper, we provide an explicit model of the large population where implications of the law of large numbers for aggregate demand and prices can be formally derived, and also show how the characterization result for a large market can be approximated with a sequence finite markets as the population size grows.

1 Introduction

Consider a uniform-price auction with a large number of traders and an exogenous supply of assets. The common value of a unit of the asset depends on an unknown state, and traders receive private signals that are *i.i.d.* conditional on the state. After observing private signals, traders submit sealed bids and an auctioneer determines the market-clearing price. In this market, prices are said to aggregate information if they convey the same information about the value of the asset as would be obtained if all signals were public.

For environments with a finite states and signals, Siga and Mihm (2020) characterize when information aggregation is an equilibrium property in this market with a large population of atomless traders. Their model shares features of both rational expectations equilibrium (REE) and strategic auction models. Similar to REE models,

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the large population implies that traders have negligible impact on prices and individual bids only impact a trader's allocation. However, similar to strategic auction models, there is an explicit protocol for the price formation process and trader's can condition bids only on their own private information.

In this paper, we extend on the analysis in Siga and Mihm (2020) in two directions. First, following an approach in Al-Najjar (2008), we provide an explicit model of the large population where we can formally derive implications of the law of large numbers (LLN) for aggregate demand and prices. Second, we show how the characterization result for the large market can be approximated by a sequence of finite markets with a growing population of traders.

In Section 2, we describe the environment and the pricing mechanism, and review the main result in Siga and Mihm (2020). Section 3 describes the large population model, and formally derives the implications of the law of large numbers for aggregated demand and prices. Section 4 describes the equilibrium approximation via a sequence of finite markets.

2 Model

We consider a uniform-price auction with a population of traders $\mathcal{I}$ and an exogenous supply of assets. Key features of the environment and market are described below.

Environment. An environment consists of a finite set of states $\Omega = \{\omega_1, ..., \omega_M\}$ and signals $S = \{s_1, ..., s_K\}$, with a joint-probability distribution P on $\Omega \times S$. In state ω, the common value of unit of asset is $v(\omega) \geq 0$. Since the state is unknown, there is uncertainty about the value. Before submitting bids, traders independently draw a signal from the conditional distribution over signals $P_\omega \in \Delta(S)$. For simplicity, we assume that each state occurs with strictly positive probability, and that different states generate different conditional distributions. The key primitives of the environment are the value function $v : \Omega \to \mathbb{R}_+$ and information structure $\{P_\omega : \omega \in \Omega\}$.

Strategies. After receiving their private signals, traders submit bids from a compact interval $B = [0, \bar{b}]$, which contains $v(\Omega)$. A strategy for trader i is therefore a mapping $\sigma_i : S \to \Delta(B)$ from signals to Borel probability distributions over bids. For simplicity, we focus in this paper on symmetric strategies. A strategy-profile can therefore also be described by a mapping $\sigma : S \to \Delta(B)$, which describes the signal-contingent bid of each trader.

Auction mechanism. The auction format provides an explicit protocol for the price formation process whereby the realized distribution of bids determines a market-clearing price.

Given a bid-profile $a : \mathcal{I} \to B$, where $a(i)$ represents trader i's bid, the price $p(a)$ is the lowest bid at which the mass of traders willing to trade exceeds the supply of assets, and all trade occurs at this price. The set of bid-profiles is denoted $\mathcal{A} = \{a : \mathcal{I} \to B\}$, and endowed with the σ-algebra $\mathbb{A}$ generated by cylinder sets of the form $\{a : a(i) = b\}$ for some $i \in \mathcal{I}$ and $b \in B$.

A trader receives a unit of the asset if her own bid is strictly above the price and does not trade if her bid is strictly below the price. To clear the market, the auctioneer uniformly randomizes over bids equal to the price. This allocation rule ensures that the market clears, treats market participants symmetrically, and guarantees that (i) no trader wins the auction with a bid strictly below the price and (ii) no trader loses the auction with a bid strictly above the price. In state ω, the payoff for a trader is $v(\omega) - p(a)$ if she trades, and 0 otherwise.

Expected payoffs and equilibrium. A strategy-profile σ and conditional distribution over signals P_ω generate a probability measure P_ω^σ over bid-profiles in state ω. The expected payoff for type $(i, s) \in \mathcal{I} \times S$ is $\Pi_i(\sigma|s) \equiv \sum_\omega \Pi_i(\sigma|\omega) P_s(\omega)$, where $P_s(\omega)$ is the probability of state ω conditional on signal s, $\Pi_i(\sigma|\omega) \equiv \int_{\mathcal{A}} \pi_i(a|\omega) dP_\omega^\sigma$ is the expected payoff conditional on state ω, and $\pi_i(a|\omega)$ is trader i's payoff in state ω for the bid-profile a. A strategy-profile is a *(Bayes-Nash) equilibrium* if each type maximizes the expected payoff given the strategy of other types.[1]

Information Aggregation. We interested in the environments where equilibrium prices can aggregate information: the prices for an equilibrium strategy convey the same information about the value of the asset as would be obtained if all signals were publicly observable. In other words, for the value of the asset, the equilibrium price is a sufficient statistic for the entire signal-profile.

[1]For the model with a large population, our result also holds if equilibrium requires only that *almost all* types are playing a best-response.

2.1 The Betweenness Property

Information aggregation requires a relationship between the value function and the information structure, which can be described in terms of a *quasi-linear function.*

Definition 1. A function $V : \Delta(S) \to \mathbb{R}$ is quasi-linear if it satisfies two conditions.

(i) *Lower semicontinuity:* all lower contour sets $\{y : V(y) \leq V(x)\}$ are closed.

(ii) *Betweenness:* $V(x) \geq V(y)$ implies $V(x) \geq V(\theta x + (1-\theta)y) \geq V(y)$.

Lower semicontinuity describes how function values can be approximated (from below, but not necessarily from above). Betweenness is the substantive condition. It implies that lower contour sets, level sets and upper contour sets are all convex. In particular, the boundaries of these sets can be represented by hyperplanes. An important special case are linear functions, where $V(x) = \alpha \cdot x$ for some $\alpha \in \mathbb{R}^K$. While linear functions are those (continuous) functions that are both concave and convex, quasi-linear functions are those (lower semicontinuous) functions that are both quasi-concave and quasi-convex. We say that an environment satisfies the *betweenness property* if there is a quasi-linear function on $\Delta(S)$ that is monotone in values:

Definition 2. An environment satisfies the betweenness property if there is a quasi-linear function $V : \Delta \to \mathbb{R}$ such that $v(\omega) > v(\omega')$ implies $V(P_\omega) > V(P_{\omega'})$.

2.2 Characterization

In Siga and Mihm (2020), we show that equilibrium prices can aggregate information in an auction with a large atomless population of traders if and only if the environment satisfies the betweenness property. To simplify exposition, Siga and Mihm (2020) assume that there is a law of large numbers that determines the aggregate demand for assets given a strategy-profile. However, there are well-known challenges with measurability and the LLNs in continuum-agent models (see, e.g., Judd 1985). In this paper, we therefore specify an explicit model of the large population, where we can formally derive implications of the LLN for aggregated demand and prices. We then show how the equilibrium results for a large population can be approximated via a sequence of finite populations.

3 Large population

Following Aumann (1964), competitive market models often consider a continuum of agents endowed with a non-atomic probability measure (e.g., Lebesgue measure on $[0,1]$), which ensures that each individual trader has a negligible impact on prices. There are, however, some well-known limitations of the continuum-agent framework. First, there is a measurability problem when agents and/or nature randomize independently, which poses a challenge in strategic settings (where agents randomize) and environments with incomplete information (where nature randomizes). Second, standard laws of large numbers do not extend to a continuum of random variables, which poses a challenge when describing aggregate outcomes such as prices.

Since we are interested in aggregate outcomes (in particular, prices) for a strategic setting with incomplete information, we use the alternative population model proposed in Al-Najjar (2008). A *large population* consists of a tuple $(\mathcal{I}, \mathbb{I}, \lambda)$, where $\mathcal{I} \subset [0,1]$ is a *countable* set of traders, $\mathbb{I}$ is the *power-set* on $\mathcal{I}$, and λ is a *finitely*-additive probability measure with $\lambda(i) = 0$ for all $i \in \mathcal{I}$. As in a continuum-agent framework, $\lambda(i) = 0$ ensures that trader i has negligible impact on aggregate outcomes. However, because $\mathbb{I}$ is the power-set, there are no measurability restrictions. Moreover, when the large population is defined as suitable limit of finite populations, an exact law of large numbers applies and provides a simple characterization of aggregate behavior. We present the formal definition below and refer to Al-Najjar (2008) for a detailed discussion (including proof of existence).[2]

Proper large population. Consider a sequence $\{\mathcal{I}_n\}_{n=1}^{\infty}$ of finite subsets of $[0,1]$, where each $\mathcal{I}_n$ can be interpreted as a finite set of traders. The sequence $\{\mathcal{I}_n\}_{n=1}^{\infty}$ is *proper* if $\mathcal{I}_n \subsetneq \mathcal{I}_{n+1}$ for all n, and $\lim_{n\to\infty} \frac{|\mathcal{I}_n|}{|\mathcal{I}_{n+1}|} = 0$ (i.e., the population grows, and at an increasing rate). The following definition describes a large population as the limit of a proper sequence of finite populations.

Definition 3. The large population $(\mathcal{I}, \mathbb{I}, \lambda)$ is *proper* if there is a proper sequence of finite populations $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$ and, for any finite collection

[2] Al-Najjar (2008) provides a detailed analysis and discussion of the connection between asymptotic equilibria in finite games, equilibria in a large population game, and equilibria in a continuum-agent game. There is an error in his result relating asymptotic equilibria in finite games with the equilibria in a large population game (see Tolvanen and Soultanis, 2012). This error is inconsequential for our analysis because our approximation result in the next section is based on entirely different arguments.

$\{I'_r \in \mathcal{I} : r = 1, ..., R\}$, there exists a subsequence $\{\mathcal{I}_h\}_{h=1}^{\infty}$ of $\{\mathcal{I}_n\}_{n=1}^{\infty}$ such that $\lambda\left(I'_r\right) = \lim_{h\to\infty} \frac{|I'_r \cap \mathcal{I}_h|}{|\mathcal{I}_h|}$ for all r.

Competitive auction. For our model of a competitive auction, let $(\mathcal{I}, \mathbb{I}, \lambda)$ be a proper large population. There is a mass $\kappa \in (0, 1)$ of assets. Each trader $i \in \mathcal{I}$ has unit demand for the asset, and a bid $b \in B$ represents the maximum price at which i is willing to buy.

For bid-profile $a : \mathcal{I} \to B$, the cumulative bid distribution is defined as $F_a(b) \equiv \int_{i\in\mathcal{I}} \mathbb{1}[a(i) \leq b]\lambda(di)$.[3] Since λ is not countably-additive, it is not guaranteed that F_a is right-continuous. However, for a symmetric strategy-profile, we show below that F_a is almost-surely right-continuous. To simplify exposition, we therefore describe the pricing mechanism for this case.

For $\gamma \in [0, 1]$, let $\mathcal{Q}_a(\gamma) \equiv \inf\{b \in B : \gamma \leq F_a(b)\}$ denote the γ-quantile of this cumulative distribution. The price $p(a) \equiv \mathcal{Q}_a(1 - \kappa)$ is therefore the lowest bid that ensures the mass of traders willing to buy at the price exceeds the supply of assets. All trade occurs at the price $p(a)$. A trader receives a unit of asset if her bid is strictly above the price, and does not trade if her bid is strictly below the price. To clear the market, the auctioneer uniformly randomizes over bids equal to the price. This allocation-rule defines the likelihood $w(i, a)$ that i trades. In particular, for bid-profile a with cumulative distribution functions F_a and price $p(a) = p$, the likelihood that trader i receives a unit of asset is

$$w(i, a) = \begin{cases} 1 & \text{if } a(i) > p \\ \frac{\kappa-(1-F_a(p))}{F_a(p)-\vec{F}_a(p)} & \text{if } a(i) = p,\ F_a(p) \neq \vec{F}_a(p) \\ 0 & \text{otherwise} \end{cases},$$

where all randomizations are independent. The price-rule $p : \mathcal{A} \to B$ and allocation-rule $w : \mathcal{I} \times \mathcal{A} \to [0, 1]$ ensure that, almost surely, the market clears. To illustrate, consider the cases where $F_a(p) > \vec{F}_a(p)$, where $\vec{F}_a(b) \equiv \lim_{b'\uparrow b} F_a(b')$ is the mass of

[3] For bounded function $f : \mathbb{R} \to \mathbb{R}$, the integral for a finitely-additive measure λ is defined as for countably additive measures by constructing integrals for simple functions, and taking a limit of a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ converging to f (see, e.g., Al-Najjar 2008, Section 2.3.1).

bids strictly less than b (the argument for other cases is analogous):

$$\begin{aligned}\int_{\mathcal{I}} w(i,a)\lambda(di) &= \int \left(0\mathbb{1}[a(i){<}p(a)] + \frac{\kappa-(1-F_a(p))}{F_a(p)-\vec{F}_a(p)}\mathbb{1}[a(i){=}p(a)] + 1\mathbb{1}[a(i){>}p(a)]\right)d\lambda \\ &= \frac{\kappa-(1-F_a(p))}{F_a(p)-\vec{F}_a(p)}\int \mathbb{1}[a(i){=}p(a)]\lambda(di) + \int \mathbb{1}[a(i){>}p(a)]\lambda(di) \\ &= \frac{\kappa-(1-F_a(p))}{F_a(p)-\vec{F}_a(p)}\left(F_a(p)-\vec{F}_a(p)\right) + (1-F_a(p)) = \kappa.\end{aligned}$$

In state ω, the payoff for trader i is therefore $\pi_i(a|\omega) \equiv w(i,a)(v(\omega)-p(a))$.

Aggregate demand. For a symmetric strategy-profile $\sigma : S \to \mathcal{B}$ and a distribution $x \in \Delta$, the Hahn-Kolmogorov Extension Theorem guarantees existence of a unique countably-additive measure P_x^σ on $(\mathcal{A}, \mathbb{A})$ when traders draw *i.i.d* signals from x (see Al-Najjar, 2008).

Let $F^{\sigma_i(s)}$ denote the cumulative distribution function for the strategy of type (i,s), and denote by $F^{\sigma_i} \equiv \left(F^{\sigma_i(s_1)}, ..., F^{\sigma_i(s_K)}\right)$ the vector-valued function that describes trader i's cumulative distribution for each signal. The cumulative distribution over bids for trader i depends on the strategy F^{σ_i} and the distribution over signals $x \in \Delta$, and is defined by $F_x^{\sigma_i}(b) \equiv F^{\sigma_i}(b) \cdot x$.

For a signal $s \in S$, the aggregate cumulative distribution over bids conditional on signal s is defined by $F_s^\sigma(b) \equiv \int_{\mathcal{I}} F^{\sigma(i,s)}(b)\lambda(di)$. Denote by $F^\sigma \equiv \left(F_{s_1}^\sigma, ..., F_{s_K}^\sigma\right)$ the corresponding vector-valued function that gives the aggregate cumulative distribution of traders for each signal. If traders draw *i.i.d* signals from $x \in \Delta$, the mean cumulative distribution of bids is defined by $F_x^\sigma(b) \equiv \int_{\mathcal{I}} F_x^{\sigma_i}(b)\lambda(di)$. Note that $F_x^\sigma = F^\sigma \cdot x$ because

$$\begin{aligned}F_x^\sigma(b) &\equiv \int_{\mathcal{I}} F_x^{\sigma_i}(b)\,\lambda(di) \equiv \int_{\mathcal{I}} [F^{\sigma_i}(b)\cdot x]\,\lambda(di) = \int_{\mathcal{I}}\left[\sum_{k=1}^K F^{\sigma_i(s_k)}(b)x(s_k)\right]\lambda(di) \\ &= \sum_{k=1}^K x(s_k)\left[\int_{\mathcal{I}} F^{\sigma_i(s_k)}(b)\lambda(di)\right] \equiv \sum_{k=1}^K P_\omega(s_k)F_{s_k}^\sigma(b) = F^\sigma(b)\cdot x.\end{aligned}$$

Lemma 1. *For a symmetric stragegy-profile σ, F_s^σ is right-continuous for all $s \in S$ and F_x^σ is right-continuous for all $x \in \Delta(S)$.*

Proof. Fix some signal $s \in S$ and bid $b \in B$. For trader i, the cumulative bid distribution $F^{\sigma_i}(s)$ is derived from a Borel distribution over bids, hence right-continuous. As a result, for a symmetric strategy-profile σ, there exists $\delta(\varepsilon) > 0$ for any $\varepsilon > 0$ such that $b' \in [b, b + \delta(\varepsilon)]$ implies $F^{\sigma_i(s)}(b') - F^{\sigma_i(s)}(b) \leq \varepsilon$ for all $i \in \mathcal{I}$. Therefore, $F^{\sigma}_s(b') - F^{\sigma}_s(b) = \int_{\mathcal{I}} \left(F^{\sigma_i(s)}(b') - F^{\sigma_i(s)}(b)\right) \lambda(di) \leq \int_{\mathcal{I}} \varepsilon \lambda(di) = \varepsilon$. That F^{σ}_x is right-continuous then follows immediately. □

The following lemma establishes implications of the strong law of large numbers in this framework: given a symmetric strategy-profile σ, the ex-post cumulative bid distribution F_a is almost surely equal to the ex-ante cumulative bid distribution F^{σ}_x for any distribution over signals $x \in \Delta(S)$.

Proposition 1. *Consider a strategy profile σ and a countable collection of bids $\{b_j\}_{j=1}^{\infty}$ in B. For any $x \in \Delta(S)$, there is a measurable subset of bid-profiles $A \in \mathcal{A}$ such that $P^{\sigma}_x(A) = 1$ and, for all $a \in A$ and $j \geq 1$, $F_a(b_j) = F^{\sigma}_x(b_j)$ and $\vec{F}_a(b_j) = \vec{F}^{\sigma}_x(b_j)$.*

The key step in the proof of Proposition 1 follows closely the arguments in the proof of Theorem 1 in Al-Najjar (2008), and we provide details only for completeness. Before providing the proof, we require some additional notation. A collection $\mathcal{U}$ of subsets of $\mathbb{N}$ is called a *free ultrafilter* if (i) $A \in \mathcal{U}$ and $A \subset B$ implies $B \in \mathcal{U}$, (ii) $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$, (iii) $\emptyset \notin \mathcal{U}$, (iv) no finite subset of $\mathbb{N}$ belongs to $\mathcal{U}$, (v) for every $A \subset \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N}/A \in \mathcal{U}$. For a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$, $\mathcal{U}$-convergence is defined by

$$\mathcal{U}\text{-}\lim a_n = a \quad \Leftrightarrow \quad \forall \varepsilon > 0,\ \{n : |a_n - a| < \varepsilon\} \in \mathcal{U}.$$

The concept of $\mathcal{U}$-convergence generalizes the usual notion of convergence; that is, if $\lim_{n\to\infty} a_n$ exists, then $\mathcal{U}\text{-}\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n$. However, every bounded sequence $\{a_n\}_{n=1}^{\infty}$ has a unique $\mathcal{U}$-limit.

The next Lemma follows from Proposition A.1 and Lemma A.2 in Al-Najjar (2008).

Lemma 2. *Let $(\mathcal{I}, \mathbb{I}, \lambda)$ be proper large population. Then there exists a free ultrafilter $\mathcal{U}$ such that, for any $I \subset \mathcal{I}$, $\lambda(I) = \mathcal{U}\text{-}\lim \frac{\#(I \cap I_n)}{\# I_n}$, and, for any bounded function $f : \mathcal{I} \to \mathbb{R}$, $\int f d\lambda = \mathcal{U}\text{-}\lim \frac{1}{\# I_n} \sum_{i=1}^{n} f(i)$.*

We are now able to provide the proof of Proposition 1.

Proof. Fix a strategy-profile σ and a distribution over signals $x \in \Delta(S)$. First fix some bid $b \in B$. For each trader $i \in \mathcal{I}$, the strategy-profile σ and conditional probability distribution x induce a distribution over bids denoted by $F_x^{\sigma_i}$. We denote the corresponding random variable by B_i. Now fix an enumeration of $\mathcal{I}$ (this is possible because $\mathcal{I}$ is countable), and consider the sequence of random variables $\{\mathbb{1}[B_1 \leq b, 1 \in \mathcal{I}], \mathbb{1}[B_2 \leq b, 2 \in \mathcal{I}], ...\}$. As traders randomize independently, this is a sequence of independent random variables on $(B, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra on B, with uniformly bounded variance and expected values

$$\mathbb{E}[\mathbb{1}[B_i \leq b, i \in \mathcal{I}]] = F_x^{\sigma_i}(b).$$

Define $F_{a,n}(b) \equiv \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[a(i) \leq b, i \in \mathcal{I}]$ and $F_{\sigma,n}(b) \equiv \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[i \in \mathcal{I}]F_x^{\sigma_i}(b)$. By Kolmogorov's Strong Law of Large Numbers (see, e.g., Shiryaev, 1984, Theorem 2, p.388), there exists $A \in \mathcal{A}$ with $P_x^{\sigma}(A)$ such that $\lim_{n\to\infty} F_{a,n}(b) = \lim_{n\to\infty} F_{\sigma,n}(b)$ for all $a \in A$.

Now fix some $a \in A$. For $\varepsilon > 0$, by the SLLN, $\left\{n : |F_{a,n}(b) - F_{\sigma,n}(b)| \leq \frac{\varepsilon}{2}\right\} \in \mathcal{U}$, where $\mathcal{U}$ is the free ultrafilter from Lemma 2 (recall that standard convergence implies $\mathcal{U}$-convergence). Moreover, since the sequence $\{F_{a,n}(b)\}_{n=1}^{\infty}$ is bounded, it follows that $\left\{n : |F_{a,n}(b) - \mathcal{U}\text{-}\lim F_{a,n}(b)| \leq \frac{\varepsilon}{2}\right\} \in \mathcal{U}$, because every bounded sequence is $\mathcal{U}$-convergent. By property (ii) of $\mathcal{U}$,

$$\left\{n : |F_{a,n}(b) - F_{\sigma,n}(b)| \leq \frac{\varepsilon}{2}\right\} \cap \left\{n : |F_{a,n}(b) - \mathcal{U}\text{-}\lim F_{a,n}(b)| \leq \frac{\varepsilon}{2}\right\} \in \mathcal{U}.$$

By the triangle inequality,

$$\begin{aligned} \left\{n : |F_{a,n}(b) - F_{\sigma,n}(b)| \leq \frac{\varepsilon}{2}\right\} \quad &\cap \quad \left\{n : |F_{a,n}(b) - \mathcal{U}\text{-}\lim F_{a,n}(b)| \leq \frac{\varepsilon}{2}\right\} \\ &\subset \quad \{n : |F_{\sigma,n}(b) - \mathcal{U}\text{-}\lim F_{a,n}(b)| \leq \varepsilon\}. \end{aligned}$$

By property (i) of an ultra filter, $\left\{n : |F_{\sigma,n}(b) - \mathcal{U}\text{-}\lim F_{a,n}(b)| \leq \frac{\varepsilon}{2}\right\} \in \mathcal{U}$. It therefore follows that $\mathcal{U}\text{-}\lim F_{\sigma,n}(b) = \mathcal{U}\text{-}\lim F_{a,n}(b)$. Hence, by Lemma 2,

$$F_a(b) \equiv \int_{\mathcal{I}} \mathbb{1}[a(i) \leq b, i \in \mathcal{I}]\, d\lambda = \int_{i\in\mathcal{I}} F_{\omega}^{\sigma_i}(b) d\lambda \equiv F_{\omega}^{\sigma}(b).$$

Now let $\{b_h\}_{h=1}^{\infty}$ be a sequence of bids such that $b_h \uparrow b_j$ for some bid $b_j \in B$. By the preceding arguments, for every h there exists a set $A_{x,h}^{j} \in \mathbb{A}$ such that $P_x^{\sigma}(A_{\omega,h}^{j}) = 1$ and $F_a(b_h) = F_{\omega}^{\sigma}(b_h)$ for all $a \in A_{x,h}^{j}$. Let $A_x^j = \bigcap_h A_{x,h}^j$. Then A_x^j is the countable intersection of measure 1 sets, and so $P_x^{\sigma}(A_x^j) = 1$ (because P_x^{σ} is countably-additive). By definition, for all $a \in A_x^j$, $F_a(b_h) = F_x^\sigma(b_h)$ for every $h \geq 1$. As a result, for all $a \in A_x^j$, $\vec{F}_a(b) \equiv \lim_{h\to\infty} F_a(b_h) = \lim_{h\to\infty} F_x^\sigma(b_h) \equiv F_x^\sigma(b)$.

Finally, let $\{b_j\}_{j=1}^\infty$ be any countable collection of bids in B. By the previous argument, for each $j \geq 1$, there exists a set $A_x^j \in \mathbb{A}$ such that $P_x^\sigma(A_x^j) = 1$ and, for all $a \in A_x^j$, $F_a(b_j) = F_x^\sigma(b_j)$ and $\vec{F}_a(b_j) = \vec{F}_x^\sigma(b_j)$. Let $A_x = \bigcap_j A_x^j$. Then A_x is the countable intersection of measure 1 sets, and so $P_x^\sigma(A_x) = 1$. By definition, for all $a \in A_x$, $F_a(b_j) = F_x^\sigma(b_j)$ and $\vec{F}_a(b_j) = \vec{F}_x^\sigma(b_j)$ for every $j \geq 1$. □

It follows from Lemma 1 that we can interpret F_x^σ as the aggregate demand for assets given a distribution over signals $x \in \Delta(S)$.

Auction pricing-function. For $\gamma \in (0,1)$, $x \in \Delta(S)$ and a symmetric strategy-profile σ, we denote the γ-quantile of F_x^σ by $\mathcal{Q}_x^\sigma(\gamma) \equiv \inf\{b \in B : \gamma \leq F_x^\sigma(b)\}$. Since F_x^σ is right-continuous, the infimum in $\mathcal{Q}_x^\sigma(\gamma)$ is attained. We can thereby define an *extended price-function* $\tilde{p}_\sigma : \Delta \to B$ by $\tilde{p}_\sigma(x) = \mathcal{Q}_x^\sigma(1-\kappa)$ for any symmetric strategy-profile σ. The extended-price function is defined for any distribution over signals, but we are primarily in those conditional distributions that are in the information structure of the environment. For a state $\omega \in \Omega$, we therefore let $p_\sigma(\omega) \equiv \tilde{p}_\sigma(P_\omega)$.

The following proposition shows that, for any distribution over signals x, $\tilde{p}_\sigma(x)$ is almost surely equal to the market clearing price $p(a)$.

Proposition 2. *If signals are* i.i.d *from $x \in \Delta(S)$, then $p(a) = \tilde{p}_\sigma(x)$ a.s.*

Proof. We show that, for any $\gamma \in (0,1)$, there is a measurable subset of bid-profiles $A_x \subset \mathcal{A}$ such that $P_x^\sigma(A_x) = 1$ and, for all $a \in A_x$, $\mathcal{Q}_a(\gamma) = \mathcal{Q}_\omega^\sigma(\gamma)$.

For $\varepsilon > 0$, let $A_\varepsilon^+ = \{a \in \mathcal{A} : \mathcal{Q}_a(\gamma) > \mathcal{Q}_x^\sigma(\gamma) + \varepsilon\}$, and let $b_\varepsilon^+ = \mathcal{Q}_x^\sigma(\gamma) + \frac{\varepsilon}{2}$. For every $a \in A_\varepsilon^+$, $\mathcal{Q}_a(\gamma) > b_\varepsilon^+$, and therefore $F_a(b_\varepsilon^+) < \gamma$. On the other hand, $\mathcal{Q}_x^\sigma(\gamma) < b_\varepsilon^+$, and therefore $F_x^\sigma(b_\varepsilon^+) \geq \gamma$. Hence, $F_a(b_\varepsilon^+) \neq F_x^\sigma(b_\varepsilon^+)$. By Lemma 1, there is a set $\tilde{A}_\varepsilon^+$ such that $P_x^\sigma\left(\tilde{A}_\varepsilon^+\right) = 1$ and $F_a(b_\varepsilon^+) = F_x^\sigma(b_\varepsilon^+)$. Hence, $\tilde{A}_\varepsilon^+ \cap A_\varepsilon^+ = \emptyset$, and so $\mathcal{Q}_a(\gamma) \leq \mathcal{Q}_x^\sigma(\gamma) + \varepsilon$ for all $a \in \tilde{A}_\varepsilon^+$.

Now let $A_\varepsilon^- = \{a \in \mathcal{A} : \mathcal{Q}_a(\gamma) < \mathcal{Q}_x^\sigma(\gamma) - \varepsilon\}$, and let $b_\varepsilon^- = \mathcal{Q}_x^\sigma(\gamma) - \frac{\varepsilon}{2}$. For every $a \in A_\varepsilon^-$, $\mathcal{Q}_a(\gamma) < b_\varepsilon^-$, and therefore $F_a(b_\varepsilon^-) \geq \gamma$. On the other hand, $\mathcal{Q}_x^\sigma(\gamma) > b_\varepsilon^-$, and therefore $F_x^\sigma(b_\varepsilon^-) < \gamma$. Again, by Lemma 1, there is a set $\tilde{A}_\varepsilon^-$ such that $P_x^\sigma\left(\tilde{A}_\varepsilon^-\right) = 1$, $\tilde{A}_\varepsilon^- \cap A_\varepsilon^- = \emptyset$, and so $\mathcal{Q}_a(\gamma) \geq \mathcal{Q}_x^\sigma(\gamma) + \varepsilon$ for all $a \in \tilde{A}_\varepsilon^-$.

Let $\tilde{A}_\varepsilon = \tilde{A}_\varepsilon^+ \cap \tilde{A}_\varepsilon^-$. Then $\tilde{A}_\varepsilon$ is the intersection of two measure 1 sets and so $P_x^\sigma\left(\tilde{A}_\varepsilon\right) = 1$. Moreover, $\mathcal{Q}_a(\gamma) \in [\mathcal{Q}_x^\sigma(\gamma) - \varepsilon, \mathcal{Q}_x^\sigma(\gamma) + \varepsilon]$ for all $a \in \tilde{A}_\varepsilon$.

Now fix a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that $\varepsilon_j \downarrow 0$. By the preceding argument, there exists a sequence $\left\{\tilde{A}_{\varepsilon_j}\right\}_{j=1}^{\infty}$ measurable such that, for every $j \geq 1$, $P_x^\sigma\left(\tilde{A}_{\varepsilon_j}\right) = 1$, and $\mathcal{Q}_a(1-g) \in [\mathcal{Q}_x^\sigma(\gamma) - \varepsilon_j, \mathcal{Q}_x^\sigma(\gamma) + \varepsilon_j]$ for all $a \in \tilde{A}_{\varepsilon_j}$. Let $A_x = \bigcap_{j=1}^{\infty} \tilde{A}_{e_j}$. Then, A_x is the intersection of a countable collection of measure 1 sets, and so $P_x^\sigma(A_x) = 1$ (because P_x^σ is countably additive). Moreover, because $\bigcap_{j=1}^{\infty} [\mathcal{Q}_x^\sigma(\gamma) - \varepsilon_j, \mathcal{Q}_x^\sigma(\gamma) + \varepsilon_j] = \{\mathcal{Q}_x^\sigma(\gamma)\}$, we have $\mathcal{Q}_a(\gamma) = \mathcal{Q}_x^\sigma(\gamma)$ for all $a \in A_x$. □

Information Aggregation. By the LLN, the proportion of traders who receive signal s in state ω is almost surely equal to $P_\omega(s)$. Public signals therefore reveal the value almost surely, and prices conveys the same information if only if there is a one-to-one mapping between values and prices.

Definition 4. Strategy-profile σ *aggregates information* if $v(\omega) \neq v(\omega')$ implies $p_\sigma(\omega) \neq p_\sigma(\omega')$.

The following theorem characterizes the environments where information aggregation is an equilibrium property of the auction mechanism.

Theorem 1. *There is a symmetric equilibrium strategy-profile that aggregates information if and only if the betweenness property is satisfied.*

Given the characterization of aggregate demand in Proposition 1 and the extended pricing-function in Proposition 2, the proof follows directly from Theorem 1 in Siga and Mihm (2020). The key difference in the current framework is that we have formally derived implications of the LLN for aggregate demand and prices with an explicit model of the large population.

4 Finite approximation

We now consider how the characterization result for the large market can be approximated via a sequence of finite markets as the size of the population grows.

Consider an increasing sequence of finite populations indexed by $n = 1, ..., \infty$. We assume that the total supply of assets is a constant share of the population size $\kappa \in (0, 1)$. Nature chooses state ω and, in each population, traders draw independent signals from the conditional distribution P_ω. Given signals, traders submit bids and the auctioneer determines the market-clearing price. In particular, for the finite population $\mathcal{I}_n$ with a quantity of assets $\mathcal{K}_n$, such that $\frac{\mathcal{K}_n}{|\mathcal{I}_n|} = \kappa \in (0,1)$, we denote a bid-profile by $a_n : \mathcal{I}_n \to B$. Let $\tilde{F}_{a_n}(b) \equiv \sum_{i \in \mathcal{I}_n} \mathbb{1}[a(i) \leq b]$ denote the number of traders submitting a bid less than or equal to b. The price is equal to $\mathcal{K}_n + 1$-highest bid. Every trader with a bid strictly greater than the price trades at the price $p(a_n)$. The auctioneer randomizes uniformly over bids equal to the price to clear the market. This defines an allocation rule $w(i, a_n)$ analogous to the large market. Let $F_{a_n}(b) \equiv \frac{\tilde{F}_{a_n}(b)}{\kappa|\mathcal{I}_n|}$ denote the normalized cumulative distribution of bids. Then $p(a_n) = \inf\{b \in B : \kappa \leq F_{a_n}(b)\}$, and the infimum is attained.

We denote by $\{\sigma_n\}_{n=1}^{\infty}$ a sequence of strategy-profiles, where σ_n is a strategy for the n-th population. A strategy-profile σ_n and distribution over signals P_ω generate a distribution over bid-profiles $P_\omega^{\sigma_n}$ in state ω, and a corresponding random price denoted $p(\sigma_n, \omega)$. By the Weak Law of Large Numbers (WLLN), the proportion of traders receiving signal s in state ω converges in probability to the conditional probability $P_\omega(s)$. Public signals therefore provide arbitrarily precise information about the value as $n \to \infty$. Accordingly, a sequence of strategy-profiles aggregates information asymptotically if the random prices eventually provide arbitrarily precise information about the value.

Definition 5. The sequence of strategy-profiles $\{\sigma_n\}_{n=1}^{\infty}$ *aggregates information asymptotically* if there is a price-function $p_{\sigma_\infty} : \Omega \to B$ such that (i) $v(\omega) \neq v(\omega')$ implies $p_{\sigma_\infty}(\omega) \neq p_{\sigma_\infty}(\omega')$, and (ii) in state ω, the sequence of prices $\{p(\sigma_n, \omega)\}_{n=1}^{\infty}$ converges in probability to $p_{\sigma_\infty}(\omega)$.[4]

We are again interested in strategy-profiles where traders respond to arbitrage opportunities. For a strategy-profile σ_n, let $\Pi_i(\sigma_n|s)$ denote the expected payoff of a type $(i,s) \in \mathcal{I}_n \times S$, and $\Pi_i^*(\sigma_n|s)$ denote the expected payoff if type (i,s) were to play a best-response. Then σ_n is an *ε-equilibrium* if $\Pi_i(\sigma_n|s) \geq \Pi_i^*(\sigma_n|s) - \varepsilon$ for all types. A 0-equilibrium is a standard Bayes-Nash equilibrium; ε-equilibrium allows for profitable deviations bounded by ε. A sequence of strategy-profiles approximates equilibrium if these bounds are allowed to vanish as the population size increases.

Definition 6. A sequence of strategy-profiles $\{\sigma_n\}_{n=1}^{\infty}$ *approximates equilibrium* if there is a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \to 0$ such that, for all n, σ_n is a ε_n-equilibrium.

[4]Formally, for $\varepsilon>0$ there is n_ε so that $P_\omega^{\sigma_n}(p(\sigma_n,\omega)\in[p_{\sigma_\infty}(\omega)-\varepsilon, p_{\sigma_\infty}(\omega)+\varepsilon]) \geq 1-\varepsilon$ when $n \geq n_\varepsilon$.

For our characterization of finite auctions we again focus on symmetric strategy-profiles. Moreover, we focus on a sequence of symmetric strategies that is population-invariant: there is some $\theta : S \to \mathcal{B}$ such that, for every n, $\theta(s) = \sigma_n(j, s)$ for all $j \in \mathcal{I}_n$. Invariant symmetric strategies are important to approximate perfectly competitive behavior: for a symmetric invariant strategy-profile, the likelihood that an individual trader can influence the price vanishes as $n \to \infty$, which is clearly not the case otherwise. For an invariant sequence of symmetric strategy profiles, the following proposition shows that the betweenness property is necessary and sufficient to aggregate information asymptotically.

Proposition 3. *There is an invariant sequence of symmetric strategy-profiles that approximates equilibrium and aggregates information asymptotically if and only if the betweenness property is satisfied.*

Before providing the proof of Theorem 3, we establish some properties of the limiting distribution of prices. Given $\theta : S \to \mathcal{B}$, let F_s^θ denote the cumulative distribution over bids for traders with signal s, and let F^θ be the corresponding vectors function. For a state ω, $F_\omega^\theta \equiv F^\theta \cdot P_\omega$. For a trader $i \in \mathcal{I}_1$ and $\theta_i : S \to \mathcal{B}$, we denote by $\{\sigma_n^{\theta_i}\}_{n=1}^\infty$ the sequence of strategy profiles where, for every n, $\sigma_n^{\theta_i}(j, s) = \theta(s)$ for $j \neq i$ and $\sigma_n^{\theta_i}(i, s) = \theta_i(s)$. Hence, $\sigma_n^{\theta_i}$ is the strategy-profile where other traders follow the symmetric strategy θ and i deviates to the strategy θ_i.

The following lemma characterizes the limiting distribution of prices using the de Moivre-Laplace central limit theorem (see, e.g., Shiryaev (1984) pp. 62-63). It also shows that the limiting distribution does not depend on the strategy-profile followed by an individual trader; that is, price-impact vanishes.

Lemma 3. *Let $\{\sigma_n\}_{n=1}^\infty$ be an invariant sequence of symmetric strategy-profiles described by $\theta : S \to \mathcal{B}$. Fix a trader $i \in \mathcal{I}_1$, a deviation θ_i for trader i, a state $\omega \in \Omega$, and a bid $b \in B$. (i) If $F_\omega^\theta(b) \leq \kappa$, then $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}(p(\sigma_n, \omega) > b) \in \left\{\frac{1}{2}, 1\right\}$, and equal to 1 if and only if $F_\omega^\theta(b) < \kappa$. (ii) If $\vec{F}_\omega^{\sigma_i}(b) \geq \kappa$, then $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}(p(\sigma_n, \omega) \leq b) \in \left\{\frac{1}{2}, 1\right\}$, and equal to 1 if and only if $\vec{F}_\omega^{\sigma_n}(b) > \kappa$.*

Proof. We show the argument for part (i), the argument for part (ii) is symmetric. For part (i), if $F_\omega^\sigma(b) = 0$, then for n sufficiently large such that $\frac{1}{|\mathcal{I}_n|} < \kappa$, we have $P_\omega^{\sigma_n^{\theta_i}}\left(\kappa \leq \tilde{F}_{a_n}(b)\right) = 0$ even when $\theta_i(s) = \delta_0$ for all s; hence $P_\omega^{\sigma_n}(p(\sigma_n, \omega) > b) = 1$. We can therefore focus on the case where $0 < F_\omega^\theta(b) \leq \kappa$. Moreover, it is without loss of generality to assume that $\theta_i(s) = \delta_{\tilde{b}}$ for some $\tilde{b} \in B$: if convergence is established for all bids $\tilde{b}$, it holds for any distribution over bids.

Let f be a generic realization of the random variable $\tilde{F}_{a_n}(b)$. For a bid-profile a_n, the price is strictly greater than b if and only if $\tilde{F}_{a_n}(b) \leq \kappa|\mathcal{I}_n| - 1$. Therefore,

$$P_\omega^{\sigma_n}(p(\sigma_n,\omega) > b) = \begin{cases} \sum_{f=0}^{\kappa|\mathcal{I}_n|-2} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1-F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} & \text{if } \tilde{b} \leq b \\ \sum_{f=0}^{\kappa|\mathcal{I}_n|-1} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1-F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} & \text{if } \tilde{b} > b \end{cases}.$$

By the de Moivre-Laplace central limit theorem, for n sufficiently large,

$$\sum_{f=0}^{\kappa|\mathcal{I}_n|-2} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1-F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} \approx \Theta\left(\frac{\kappa|\mathcal{I}_n| - 2 - (|\mathcal{I}_n|-1)F_\omega^\theta(b)}{\sqrt{(|\mathcal{I}_n|-1)F_\omega^\theta(b)(1-F_\omega^\theta(b))}}\right),$$

where Θ is the cumulative distribution function of the standard normal distribution. Likewise,

$$\sum_{f=0}^{\kappa|\mathcal{I}_n|-1} \binom{|\mathcal{I}_n|-1}{f} F_\omega^\theta(b)^f (1-F_\omega^\theta(b))^{|\mathcal{I}_n|-1-f} \approx \Theta\left(\frac{\kappa|\mathcal{I}_n| - 1 - (|\mathcal{I}_n|-1)F_\omega^\theta(b)}{\sqrt{(|\mathcal{I}_n|-1)F_\omega^\theta(b)(1-F_\omega^\theta(b))}}\right).$$

As a result, $\lim_{n\to\infty} P_\omega^{\sigma_n}(p(\sigma_n,\omega)>b) = \frac{1}{2}$ if $F_\omega^\theta(b)=\kappa$, and $\lim_{n\to\infty} P_\omega^{\sigma_n}(p(\sigma_n,\omega)>b) = 1$ if $F_\omega^\theta(b) < \kappa$. □

As a corollary of Lemma 3, we characterize when the prices converge to values.

Corollary 1. *Let $\{\sigma_n\}_{n=1}^\infty$ be an invariant sequence of symmetric strategy-profiles described by $\theta : S \to \mathcal{B}$. Fix a trader $i \in \mathcal{I}_1$, a deviation θ_i for trader i, and a state $\omega \in \Omega$. The sequence of prices $\{p(\sigma_n,\omega)\}_{n=1}^\infty$ converges in probability to $v(\omega)$ if and only if $F_\omega^\theta(v(\omega)-\delta) < \kappa < F_\omega^\theta(v(\omega)+\delta)$ for every $\delta > 0$.*

Proof. (1) Suppose that $F_\omega^\theta(v(\omega)-\delta) < \kappa < F_\omega^\theta(v(\omega)+\delta)$ for every $\delta > 0$. Fix some $\varepsilon > 0$: we want to show that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) \in [v(\omega)-\varepsilon, v(\omega)+\varepsilon]\right) = 1$. Because F_ω^θ is monotone non-decreasing, it has a countable number of points of discontinuity. We can therefore choose $\varepsilon' \in (0,\varepsilon]$ such that F_ω^θ is continuous at $v(\omega)+\varepsilon'$ and $v(\omega)-\varepsilon'$. As $F_\omega^\theta(v(\omega)-\varepsilon') < \kappa$, it follows by Lemma 3 that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) > v(\omega)-\varepsilon'\right) = 1$. Moreover, because $\kappa > F_\omega^\theta(v(\omega)+\varepsilon')$ $= \vec{F}_\omega^\theta(v(\omega)+\varepsilon')$, it also follows by Lemma 3 that $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) \leq v(\omega)+\varepsilon'\right)$ $= 1$. Hence, $\lim_{n\to\infty} P_\omega^{\sigma_n^{\theta_i}}\left(p(\sigma_n^{\theta_i},\omega) \in [v(\omega)-\varepsilon, v(\omega)+\varepsilon]\right) = 1$.

(2) For the converse, suppose that $F_\omega^\theta(v(\omega)+\delta) \leq \kappa$ for some $\delta > 0$. Then by Lemma 3, we have $\lim_{n\to\infty} P_\omega^{\sigma_i^n}(p_n^{\sigma_i}(\omega) > v(\omega)+\delta) \geq \frac{1}{2}$, and so the price does not converge in probability to $v(\omega)$. On the other hand, if $F_\omega^\theta(v(\omega)-\delta) \geq \kappa$ for some $\delta > 0$, then by Lemma 3, $\lim_{n\to\infty} P_\omega^{\sigma_i^n}(p_n^{\sigma_i}(\omega) \leq v(\omega)-\delta) \geq \frac{1}{2}$, and so the price does not converge to $v(\omega)$. □

We are now able to provide the proof for Proposition 3.

Proof. The first step is to show that an invariant sequence of symmetric strategy-profiles aggregates information asymptotically and approximates equilibrium if and only if the price converges in probability to the value in every state. The sufficiency part follows immediately from Lemma 3, because prices converge for any deviation by trader i; thus, expected payoffs converge to zero for any deviation. The necessity side is similar to the argument for the large market, and we therefore omit the details (see Proposition 7 in Siga and Mihm, 2020). The following steps complete the argument.

(1) Suppose the environment satisfies the betweenness property. Consider the symmetric strategy-profile $\sigma : S \to \mathcal{B}$ constructed in the proof of Theorem 1 in Siga and Mihm (2020). This strategy-profile can be described by a function $\theta : S \to \mathcal{B}$ such that $\sigma(i,s) = \theta(s)$ for all $i \in \mathcal{I}$, and $F^\theta = F^\sigma$. The strategy θ therefore has the property that, for every state ω, $F_\omega^\theta(v(\omega)-\delta) < \kappa < F_\omega^\theta(v(\omega)+\delta)$ for every $\delta > 0$. Let $\{\sigma_n\}_{n=1}^\infty$ be the symmetric sequence of strategy-profiles described by θ. By Corollary 1, the sequence of prices converges in probability to the value in every state. Hence, $\{\sigma_n\}_{n=1}^\infty$ aggregates information asymptotically and approximates equilibrium.

(2) For the converse, suppose that the symmetric sequence of strategy-profiles $\{\sigma_n\}_{n=1}^\infty$, described by $\theta : S \to \mathcal{B}$, aggregates information asymptotically and approximates equilibrium. Since prices must converge in probability to the value in every state, by Corollary 1, for every state ω, $F_\omega^\theta(v(\omega)-\delta) < \kappa < F_\omega^\theta(v(\omega)+\delta)$ for every $\delta > 0$. Since F_ω^θ is right-continuous, it follows that $\kappa \leq F_\omega^\theta(v(\omega))$. Moreover, if $v(\omega') < v(\omega)$, then $F_\omega^\theta(v(\omega')) < \kappa \leq F_\omega^\theta(v(\omega))$. Therefore, $V(x)$ defined as the $(1-\kappa)$-quantile of $F^\theta \cdot x$ is quasi-linear and monotone in values. □

5 Conclusion

The main result in Siga and Mihm (2020) characterizes the environments where equilibrium prices in a competitive auction market can aggregate information. Following an approach in Al-Najjar (2008), this paper first provides an explicit model of the large population where the implications of the law of large numbers for aggregate demand and prices can be formally derived. Second, we show that the characterization result for the large market also holds approximately in finite markets with a sufficiently large population of traders.

The results for finite markets reflects a similar economic intuition as the result with a perfectly competitive market. For a sufficiently large population, the LLN ensures that idiosyncratic noise vanishes and only the aggregate uncertainty about the state remains. As a result, the information conveyed by prices becomes increasingly precise. When prices convey precise information, competition ensures they must be close to values to prevent arbitrage opportunities. In a finite market, traders can also potentially profit by exploiting their impact on prices, but this market power vanishes as the population grows. Finally, the extended price-function converges to a quasi-linear function, and so prices converge in probability to values if and only if the betweenness property is satisfied.

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