A Behavioral Characterization of the Likelihood Ratio Order

Maximilian Mihm Lucas Siga*

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Abstract

It is well known that stochastic dominance is equivalent to a unanimity property for monotone expected utilities. For lotteries over a finite set of prizes, we establish an analogous relationship between likelihood-ratio dominance and monotone betweenness preferences, which are an important generalization of expected utility.

Key words: betweenness preferences, expected utility, likelihood-ratio dominance, stochastic dominance.

Consider the set of lotteries over a finite set of monetary prizes. In a seminal paper, Quirk and Saposnik (1962) show that lottery p (first-order) stochastically dominates lottery q if and only if *every* monotone expected utility function assigns a higher utility to p than q.[1] The significance of this result is two-fold. First, it provides a behavioral interpretation of stochastic dominance, which is a prominent stochastic order in probability and statistics. Second, it facilitates non-parametric predictions for behavior under risk, as commonly encountered in economics and finance: if more money is preferred to less, stochastic dominance characterizes the observable implications of the expected utility hypothesis.

In this paper, we establish an analogous relationship between *likelihood-ratio dominance*, which is another prominent stochastic order in probability and statistics, and *betweenness preferences*, which are an important generalization of expected utility introduced in Chew (1983) and Dekel (1986).

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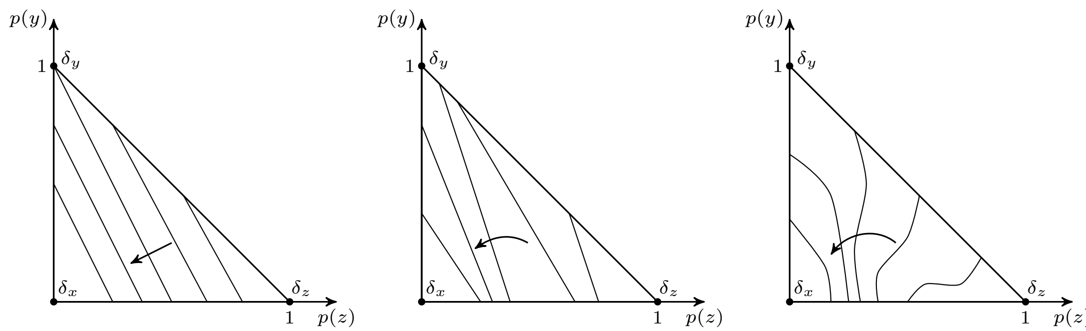
*Mihm: Division of Social Science, New York University Abu Dhabi. Email: max.mihm@nyu.edu. Siga: Division of Social Science, New York University Abu Dhabi. Email: lucas.siga@nyu.edu. We thank Nageeb Ali, Ricardo Alonso, Andrea Attar, Pablo Beker, Larry Blume, Madhav Chandrasekher, Navin Kartik, Vijay Krishna, Efe Ok, Larry Samuelson, Tom Sargent, Joel Sobel, Simone Cerreia Vioglio, and various conference and seminar audiences for helpful feedback. We also thank four anonymous referees for very helpful comments.

[1] In another seminal paper, Hadar and Russell (1969) show that the characterization of stochastic dominance also holds for continuous random variables. Similar to Quirk and Saposnik (1962), we focus on settings with a finite set of prizes in this paper.

Betweenness preferences are characterized by weakening the controversial independence axiom and can thereby accommodate widely-documented violations of the expected utility hypothesis (e.g., the Allais, 1953, paradox). Similar to expected utility, the indifference curves of a betweenness preference are linear but, unlike expected utility, are not necessarily parallel (see Figure 1). As such, betweenness preferences can remedy descriptive failures of the expected utility hypothesis while maintaining both quasiconvexity and quasiconcavity. These properties are analytically attractive because, for instance, quasiconcavity is required to establish existence of Nash equilibrium, while quasiconvexity is necessary for dynamic consistency in intertemporal choice (see, e.g., Dekel, 1986).



(a) Expected utility (b) Betweenness (c) Non-betweenness

Figure 1: Indifference curves for monotone preferences.

Consider three prizes with a given order $x \trianglerighteq y \trianglerighteq z$*. The probability of prize* z *(*y*) is measured on the horizontal (vertical) axis. A preference relation* $\succeq$ *over lotteries is monotone for the order* $\trianglerighteq$ *over prizes if* $\delta_x \succeq \delta_y \succeq \delta_z$*. Panel (a) illustrates a monotone expected utility: indifference curves are straight and parallel. Panel (b) illustrates a monotone betweenness preference: indifference curves are straight but not parallel. Panel (c) represents a monotone preference that is not in the class of betweenness preferences: indifference curves are not straight.*

As with the classic relationship between stochastic dominance and expected utility, our characterization serves a dual purpose. For decision analysis, it provides a simple criterion for making robust predictions across the whole class of betweenness preferences based on a well-known stochastic order from probability and statistics. Conversely, the characterization provides a behavioral foundation for the likelihood-ratio order that may prove useful when interpreting results from information economics and mechanism-design, where the likelihood-ratio order is often a central assumption. We discuss these interpretations in more detail after presenting the result.

The paper is organized as follows. Section I describes monotone betweenness preferences in terms of their axiomatic properties. Section II recalls the stochastic and likelihood-ratio orders, and illustrates the connection to monotone preferences geometrically. In Section III, we state, prove and discuss the characterization result.

I Betweenness preferences

Let Δ be the set of probability distributions over a finite set X. Typical elements of X are denoted x, y, z and called *prizes*. Typical elements of Δ are denoted p, q, r and called *lotteries*. We denote by $p(x)$ the probability that lottery p assigns to prize x, and let δ_x be the lottery that assigns probability 1 to prize x. The set of prizes is endowed with a total order $\trianglerighteq$, which is a primitive ranking of the prizes. For instance, if prizes are monetary, it may be natural to define $x \trianglerighteq y$ if and only if $x \geq y$.

Let $\succeq$ be a binary relation on the set of lotteries, with asymmetric part $\succ$ and symmetric part $\sim$. The following axioms are standard in the literature on decision-making under risk, where $\succeq$ is interpreted as a preference relation over lotteries.

Axiom 1 (Weak order)**.** For all $p, q, r \in \Delta$: (i) $p \succeq q$ or $q \succeq p$, and (ii) $p \succeq q$ and $q \succeq r$ implies $p \succeq r$.

Axiom 2 (Continuity)**.** If $p \succ q \succ r$, then $\theta p + (1-\theta) r \sim q$ for some $\theta \in (0,1)$.

Axiom 3 (Non-triviality)**.** There are lotteries p and q such that $p \succ q$.

The binary relation $\succeq$ has an expected utility representation if and only if, in addition to Axioms 1–3, $\succeq$ satisfies the independence axiom:

Axiom 4 (Independence)**.** For all $\theta \in (0,1)$, $p \succeq q$ implies $\theta p + (1-\theta) r \succeq \theta q + (1-\theta) r$.

Motivated by empirical violations of the independence axiom, Dekel (1986) proposes the following generalization:

Axiom 5 (Betweenness)**.** For all $\theta \in (0,1)$, (i) $p \succ q$ implies $p \succ \theta p + (1-\theta) q \succ q$, and (ii) $p \sim q$ implies $p \sim \theta p + (1-\theta) q \sim q$.

It is easily verified that independence implies betweenness but not vice versa. The generalization is consequential because the betweenness axiom is consistent with behavioral phenomena that are precluded by the independence axiom, such as the Allais paradox or disappointment aversion (see, e.g., Machina, 1982; Chew, 1983; Dekel, 1986; Gul, 1991; Starmer, 2000; Cerreia-Vioglio, Dillenberger and Ortoleva, 2020).

Definition 1. The binary relation $\succeq$ is (i) a *linear preference* if it satisfies Axioms 1–4, and (ii) a *betweenness preference* if it satisfies Axioms 1–3 and Axiom 5.

We require one further property of the preference relation: a betweenness/linear preference is *monotone* if a greater prize (for sure) is preferred to a lesser prize (for sure):

Axiom 6 (Monotonicity)**.** $x \trianglerighteq y$ implies $\delta_x \succeq \delta_y$.

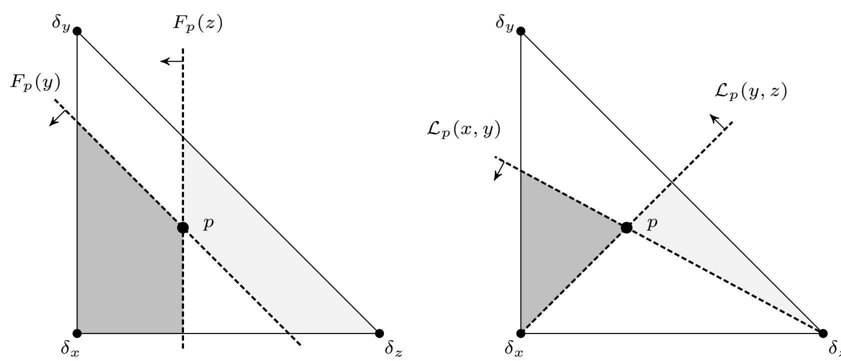
II Stochastic orders

We recall two stochastic orders over lotteries, which occupy a central role in probability and statistics (see, e.g., Shaked and Shanthikumar, 2007). For a lottery $p \in \Delta$, let $F_p(x) \equiv \sum_{y \trianglelefteq x} p(y)$ be the cumulative distribution function over prizes.

Definition 2. For lotteries p and q, (i) p *stochastically dominates* q if $F_p(x) \leq F_q(x)$ for all prizes x, and (ii) p *likelihood-ratio dominates* q if $p(x)q(y) \geq p(y)q(x)$ for all prizes $x \trianglerighteq y$.

For lotteries with full support, likelihood-ratio dominance can be equivalently expressed in terms of likelihood-ratios. For a lottery p with full support let $\mathcal{L}_p(x, y) \equiv p(x)/p(y)$ be the likelihood-ratio function. Then for lotteries p and q with full support, p likelihood-ratio dominates q if and only if $\mathcal{L}_p(x, y) \geq \mathcal{L}_q(x, y)$ for all prizes $x \trianglerighteq y$.

To illustrate these stochastic orders, suppose there are three prizes, $x \trianglerighteq y \trianglerighteq z$, so that lottery p can be depicted as a point in the unit simplex as described in Figure 1.



(a) Stochastic dominance

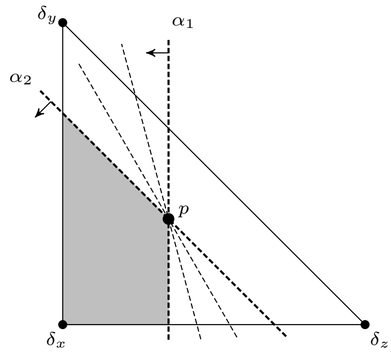
(b) Likelihood-ratio dominance

Figure 2: Geometric illustration of stochastic orders.

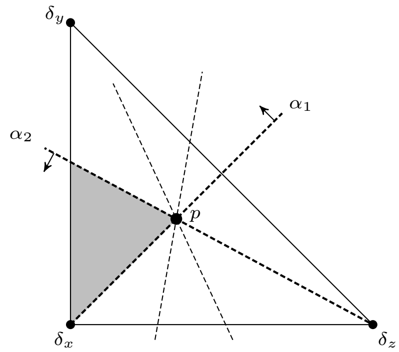
In Figure 2a, any lottery q on the line $F_p(z)$ satisfies $F_q(z) = F_p(z)$, while any lottery r on the line $F_p(y)$ satisfies $F_r(y) = F_p(y)$. As a result, the dark region represents lotteries that stochastically dominate p; the light region represents lotteries that are stochastically dominated by p; and remaining lotteries are stochastically non-comparable to p.

In Figure 2b, any lottery q on the line $\mathcal{L}_p(y, z)$ satisfies $\mathcal{L}_q(y, z) = \mathcal{L}_p(y, z)$, while any lottery r on the line $\mathcal{L}_p(x, y)$ satisfies $\mathcal{L}_r(x, y) = \mathcal{L}_p(x, y)$. The dark region therefore represents lotteries that likelihood-ratio dominate p; the light region represents lotteries that are likelihood-ratio dominated by p; and remaining lotteries are likelihood-ratio non-comparable to p. The dark (resp. light) region in Figure 2b is a strict subset of the dark (resp. light) region in Figure 2a, reflecting that likelihood-ratio dominance implies stochastic dominance.

The following figure illustrates the connection to monotone preferences, and provides much of the intuition for our result.



(a) Linear preferences



(b) Betweenness preferences

Figure 3: Geometric illustration of monotone preferences.

Figure 3a illustrates potential indifference curves containing lottery p for monotone linear preferences. Without monotonicity, any straight line can represent an indifference curve for some linear preference, but monotonicity imposes additional restrictions. Line α_1 represents an indifference curve of a monotone linear preference, as long as the upper contour set is to the left. A clockwise rotation would not represent an indifference curve because a translation has δ_y in the upper and δ_x in the lower contour set (violating $\delta_x \succeq \delta_y$). Rotating anti-clockwise yields other potential indifference curves but beyond line α_2, translations will have δ_y in the lower and δ_z in the upper contour set (violating $\delta_y \succeq \delta_z$). As a result, only straight lines between α_1 and α_2 can represent indifference curves of a monotone linear preference. The shaded region therefore represents lotteries that are preferred to p for *every* monotone linear preference, which coincides with the dark region in Figure 2a.

For a betweenness preference, indifference curves are also straight lines but do not need to be parallel, which implies that there is a larger set of potential indifference curves. In Figure 3b, line α_1 cannot be an indifference curve for a monotone betweenness preference because it contains δ_x but has δ_y in the upper contour set (violating $\delta_x \succeq \delta_y$). However, rotating anti-clockwise yields a potential indifference curve because, while a translation would have δ_y in the upper and δ_x in the lower contour set, higher indifference curves need not be parallel. As long as the line passing through p has both δ_y and δ_x in the upper contour set, it is possible to complete the map of indifference curves to represent a monotone betweenness preference (see Lemma 3). Again, one cannot rotate too far because line α_2 contains δ_z but has δ_y in the lower contour set (violating $\delta_y \succeq \delta_z$). The shaded region in Figure 3b therefore represents lotteries that are preferred to p for *every* monotone betweenness preference, which coincides with the dark region in Figure 2b.

III Characterization

The intuition from Figures 2a and 3a extends to more than three prizes: lottery p stochastically dominates lottery q if and only if $p \succeq q$ for every monotone linear preference. We establish an analogous relationship between the likelihood-ratio order and monotone betweenness preferences.

Theorem 1. *Lottery p likelihood-ratio dominates lottery q if and only if $p \succeq q$ for every monotone betweenness preference $\succeq$.*

As Figures 2b and 3b suggest, the proof is geometric. In Lemma 1, we first characterize likelihood-ratio dominance in terms of a single-crossing property of supporting hyperplanes. In Lemma 2, we then invoke a result from Siga and Mihm (2020) to show that single-crossing also characterizes monotonicity for betweenness preferences. Finally, Lemma 3 shows how to complete a map of indifference curves for a monotone betweenness preference. Theorem 1 follows immediately from these arguments.

Notation: Denote the likelihood-ratio dominance relation by $\trianglerighteq_{LR}$. Without loss of generality, enumerate prizes $X = \{x_1, ..., x_K\}$ so that $x_{k+1} \trianglerighteq x_k$ for $k = 1, .., K-1$. In the following, all vectors are in $\mathbb{R}^K$ and $\alpha(k)$ is the k-th component of vector α. We identify Δ with the unit simplex on $\mathbb{R}^K$ and set $p(k) \equiv p(x_k)$ and $\delta_k \equiv \delta_{x_k}$.

We say that a vector α is *single-crossing* if, for all $k = 1, ..., K-1$, (i) $\alpha(k+1) \leq 0$ implies $\alpha(k) \leq 0$, and (ii) $\alpha(k+1) < 0$ implies $\alpha(k) < 0$. For lottery $p \in \Delta$, let $\mathcal{C}(p)$ be the set of non-zero single-crossing vectors α such that $\alpha \cdot p = 0$.

Likelihood-ratio dominance and single-crossing: The following lemma provides a geometric characterization of likelihood-ratio dominance in terms of single-crossing vectors.

Lemma 1. *Lottery p likelihood-ratio dominates q if and only if $\alpha \cdot p \geq 0$ for all $\alpha \in \mathcal{C}(q)$.*

Proof. [**only if**] Suppose p likelihood-ratio dominates q and $\alpha \in \mathcal{C}(q)$. Since $\alpha \cdot q = 0$, $\alpha(k) \geq 0$ for some $k \in \{1, ..., K\}$. Hence, $k^+ \equiv \min\{k : \alpha(k) \geq 0\}$ is well-defined and $K^+ \equiv \{k : k \geq k^+\}$ is non-empty. If $\alpha(k) \geq 0$ for all k, then $\alpha \cdot p \geq 0$, and so we focus on the case where $K^- \equiv \{k : k < k^+\}$ is also non-empty. Single-crossing implies that $\alpha(k) < 0 \leq \alpha(k')$ whenever $k \in K^-$ and $k' \in K^+$. We proceed by looking at three cases.

Case 1: $p(k)q(k) = 0$ for all $k \in K^+$. Consider three subcases:

(a) Suppose $p(k) = q(k) = 0$ for all $k \in K^+$. In that case, $\alpha \cdot q < 0$, which contradicts $\alpha \in \mathcal{C}(q)$. As a result, $\hat{k} = \min\{k \in K^+ : p(k) + q(k) > 0\}$ is well-defined, and either $p(\hat{k}) > 0 = q(\hat{k})$ or $p(\hat{k}) = 0 < q(\hat{k})$.

(b) Suppose $p(\hat{k}) > 0 = q(\hat{k})$. Since $p \trianglerighteq_{LR} q$, $p(k)q(\hat{k}) \geq p(\hat{k})q(k)$ for all $k > \hat{k}$. As a result, $q(k) = 0$ for all $k \geq \hat{k}$ (since $p(\hat{k}) > 0$). Hence, $q(k) = 0$ for all $k \in K^+$ since $p(k) + q(k) = 0$ for $k \in \{k^+, ..., \hat{k} - 1\}$. This again contradicts that $\alpha \in \mathcal{C}(q)$.

(c) Finally, suppose $p(\hat{k}) = 0 < q(\hat{k})$. Since $p \trianglerighteq_{LR} q$, $p(\hat{k})q(k) = 0 \geq p(k)q(\hat{k})$ for all $k < \hat{k}$, which implies that $p(k) = 0$ for all $k < \hat{k}$ (since $q(\hat{k}) > 0$). In particular, $p(k) = 0$ for all $k \in K^-$, and so $\alpha \cdot p \geq 0$.

Since $p(k)q(k) = 0$ for all $k \in K^+$ implies $\alpha \cdot p \geq 0$, we focus on remaining cases where $k^* \equiv \min\{k \in K^+ : p(k)q(k) > 0\}$ is well-defined, and $K^* \equiv \{k : k \geq k^*\}$ is non-empty.

Case 2: $p(k) + q(k) > 0$ for some $k \in K^+ \backslash K^*$. Consider two subcases.

(a) Suppose $p(k) > 0 = q(k)$. Then $p(k^*)q(k) = 0 < p(k)q(k^*)$, contradicting $p \trianglerighteq_{LR} q$.

(b) Alternatively, suppose $p(k) = 0 < q(k)$. Since $p \trianglerighteq_{LR} q$, $p(k)q(k') = 0 \geq p(k')q(k)$ for all $k' < k$. As a result, $p(k') = 0$ for all $k' < k$ (since $q(k) > 0$). In particular, $p(k) = 0$ for all $k \in K^-$, and so $\alpha \cdot p \geq 0$.

Case 3: Suppose $p(k) = q(k) = 0$ for all $k \in K^+ \backslash K^*$. Since $p(k^*)q(k^*) > 0$, $p \trianglerighteq_{LR} q$ implies that $q(k) \geq (q(k^*)/p(k^*))p(k)$ for all $k < k^+$ and $q(k) \leq (q(k^*)/p(k^*))p(k)$ for all $k \geq k^*$. Since α is single-crossing, $\alpha(k) \leq 0$ for all $k < k^+$, and $\alpha(k) \geq 0$ for all $k \geq k^*$. Therefore,

$$
\begin{aligned}
(1) \qquad 0 = \alpha \cdot q &= \sum_{k=1}^{k^+-1} \alpha(k)q(k) + \sum_{k=k^*}^{K} \alpha(k)q(k) \\
&\leq \sum_{k=1}^{k^+-1} \alpha(k)\frac{q(k^*)}{p(k^*)}p(k) + \sum_{k=k^*}^{K} \alpha(k)\frac{q(k^*)}{p(k^*)}p(k) = \frac{q(k^*)}{p(k^*)}\alpha \cdot p
\end{aligned}
$$

Since $q(k^*)/p(k^*) > 0$, it follows that $\alpha \cdot p \geq 0$.

[If] For the converse, suppose $\alpha \cdot p \geq 0$ for all $\alpha \in \mathcal{C}(q)$. To establish that $p \trianglerighteq_{LR} q$, we fix $n, m \in \{1, ..., K\}$ such that $n > m$ and show that $p(n)q(m) \geq p(m)q(n)$. If $q(n) = 0$, then $p(n)q(m) \geq p(m)q(n)$. We therefore focus on $q(n) > 0$, and again consider three cases.

Case 1: Suppose $q(m) > 0$. For any $\varepsilon \in (0, 1)$ define the vector α_ε by

$$
(2) \qquad \alpha_\varepsilon(k) \equiv \begin{cases} -(1-\varepsilon)q(n) - \varepsilon \sum_{i=n+1}^{K} q(i) & \text{if } k = m \\ (1-\varepsilon)q(m) + \varepsilon \sum_{i=1}^{m-1} q(i) & \text{if } k = n \\ -\varepsilon q(n) & \text{if } k < m \\ \varepsilon\, q(m) & \text{if } k > n \\ 0 & \text{otherwise} \end{cases}
$$

Then α_ε is single-crossing and $\alpha_\varepsilon \cdot q = 0$, hence $\alpha_\varepsilon \in \mathcal{C}(q)$. Therefore,

$$0 \leq \alpha_\varepsilon \cdot p = (1-\varepsilon)\Big(q(m)p(n) - q(n)p(m)\Big) + \varepsilon\psi, \tag{3}$$

where $\psi = p(n)\sum_{i=1}^{m-1} q(i) - q(n)\sum_{i=1}^{m-1} p(i) + q(m)\sum_{i=n+1}^{K} p(i) - p(m)\sum_{i=n+1}^{K} q(i)$. Since the inequality holds for all $\varepsilon \in (0,1)$, it follows that $p(n)q(m) \geq p(m)q(n)$.

Case 2: Suppose $q(k) = 0$ for all $k \leq m$. Define the vector α' by

$$\alpha'(k) \equiv \begin{cases} -1 & \text{if } k \leq m \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

Then α' is single-crossing and $\alpha' \cdot q = 0$, hence $\alpha' \in \mathcal{C}(q)$. Therefore, $0 \leq \alpha' \cdot p \leq -p(m)$, which implies $p(m) = 0$. Therefore, $p(n)q(m) \geq p(m)q(n)$.

Case 3: Suppose $q(m) = 0$ and $\sum_{k=1}^{m-1} q(k) > 0$. For any $\phi > 0$ define the vector α_ϕ by

$$\alpha_\phi(k) \equiv \begin{cases} -\sum_{i=n}^{K} q(i) & \text{if } k < m \\ -\phi & \text{if } k = m \\ \sum_{i=1}^{m-1} q(i) & \text{if } k \geq n \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

Then α_ϕ satisfies single-crossing and $\alpha_\phi \cdot q = 0$, hence $\alpha_\phi \in \mathcal{C}(q)$. Therefore,

$$0 \leq \alpha_\phi \cdot p = \left(\sum_{i=n}^{K} p(i)\right)\left(\sum_{i=1}^{m-1} q(i)\right) - \left(\sum_{i=n}^{K} p(i)\right)\left(\sum_{i=1}^{m-1} q(i)\right) - \phi p(m). \tag{6}$$

Since the inequality holds for all $\phi > 0$, $p(m) = 0$ and so $p(n)q(m) \geq p(m)q(n)$. □

Remark 1: For lotteries with full support, the proof of Lemma 1 simplifies because only two of the six cases are relevant (Case 3 for the "only if" part, and Case 1 for the "if" part). Since the likelihood-ratio order is continuous, one could also use a limit argument to extend the characterization with full support to lotteries on the boundary of the simplex. However, the argument is not as straightforward as the analogous proof for stochastic dominance, where independence implies that one can directly consider mixtures of a boundary lottery with any interior lottery.

Remark 2: Lemma 1 provides a single-crossing characterization of the likelihood-ratio order for lotteries on a finite domain.[2] There are numerous other characterizations of the

[2] In different contexts from ours, a connection between ratio orders and single-crossing properties has been

likelihood-ratio order in the literature (see, e.g., Shaked and Shanthikumar, 2007, for a survey of results). A very general result is obtained by Lehrer and Wang (2020), who define a *strong stochastic dominance* relation for cumulative distribution functions on the real line, which is equivalent to likelihood-ratio dominance when the cumulative distribution functions admit densities.[3] Their main result characterizes strong stochastic dominance in terms of a simple property of the expected values with respect to the cumulative distribution functions.

The main result in Lehrer and Wang (2020) does not apply directly to our framework but, with an appropriate mapping from finite lotteries to cumulative distribution functions on the real line, one can show that likelihood ratio dominance on a finite domain is equivalent to strong stochastic dominance of the induced cumulative distribution functions.[4] The characterization of strong stochastic dominance in Lehrer and Wang (2020) then implies a single-crossing property, which can also be used to provide an alternative proof for the "only if" part in Lemma 1.

Monotone betweenness preferences and single-crossing: We next show that single-crossing is also directly related to the monotonicity axiom for betweenness preferences.

A binary relation $\succeq$ on Δ is represented by a function $V : \Delta \to \mathbb{R}$ if, for all lotteries,

$$p \succeq q \quad \iff \quad V(p) \geq V(q). \tag{7}$$

The function V satisfies betweenness if, for all $\theta \in (0,1)$ and lotteries p and q: (i) $V(p) > V(q)$ implies $V(p) > V(\theta p + (1-\theta)q) > V(q)$, and (ii) $V(p) = V(q)$ implies $V(p) = V(\theta p + (1-\theta)q) = V(q)$. By standard arguments (see, e.g., Dekel, 1986, Proposition A.1), a binary relation $\succeq$ on Δ is a betweenness preference if and only if it is represented by a non-constant continuous function $V : \Delta \to \mathbb{R}$ that satisfies betweenness.

Since V is continuous and Δ is compact, $V(\Delta)$ is a compact interval, and betweenness implies that V is quasi-linear (i.e., both quasi-concave and quasi-convex). As a result,

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used in the prior literature. For instance, Kartik, Lee and Rappoport (2019) show that a linear combination of vectors has a single-crossing property if and only if each vector individually has the single-crossing property and the vectors are ratio-ordered (see also Quah and Strulovici, 2012).

[3]Cumulative distribution function F strongly stochastically dominates G if and only if F is a convex transformation of G, which is known to be equivalent to likelihood-ratio dominance when densities exist (Section 1.C.1 in Shaked and Shanthikumar, 2007). Convex transformations of cumulative distribution functions also feature in prior literatures on mathematical statistics (e.g., Chan, Proschan and Sethuraman, 1990), Bayesian learning (e.g., Bikhchandani, Segal and Sharma, 1992), and comparative statics of risk-aversion with non-expected utility (e.g., Chateauneuf, Cohen and Meilijson, 2004).

[4]For instance, for a real number $y \geq 1$, let $k(y) = \max\{k \in \{1, ..., K\} : k \leq y\}$ and for a lottery $p \in \Delta$, define the cumulative distribution function $F_p^* : \mathbb{R} \to [0,1]$ by $F_p^*(y) \equiv 0$ if $y < 1$ and $F_p^*(y) = F_p\left(x_{k(y)}\right)$ if $y \geq 1$. Then one can show $p \trianglerighteq_{LR} q$ if and only if F_p^* strongly stochastically dominates F_q^* following a similar argument as the proof of Lemma 2 in Lehrer and Wang (2020).

Lemma 4 in Siga and Mihm (2020) shows that any betweenness preference $\succeq$ also has a *vector-representation*: there is a collection of non-zero vectors $\{\alpha_p\}_{p\in\Delta}$ such that $q \overset{\succsim}{\precsim} p$ if and only if $\alpha_p \cdot q \gtreqless 0$; moreover, $q \overset{\succsim}{\precsim} p$ if and only if $\alpha_p \gtreqless \alpha_q$, so that $(\{\alpha_p\}_{p\in\Delta}, \geq)$ is a chain with the standard vector order $\geq$ on $\mathbb{R}^K$.[5] The vector α_p in the vector-representation can be interpreted as the norm of a hyperplane, with constant 0, which represent the indifference curves for lottery p, and the collection of norms $\{\alpha_p\}_{p\in\Delta}$ therefore represents the map of indifference curves for the betweenness preference.[6]

The following lemma shows that single-crossing characterizes the additional implications of monotonicity (Axiom 6) for the vector-representation of a betweenness preference.

Lemma 2. *Let $\succeq$ be any betweenness preference, and let $\{\alpha_p\}_{p\in\Delta}$ be its vector-representation. Then, $\succeq$ satisfies Axiom 6 if and only if α_p is single-crossing for all lotteries p.*

Proof. Suppose that $\succeq$ satisfies Axiom 6. Consider any lottery p and suppose $\alpha_p(k) \leq 0$. Then, $\alpha_p \cdot \delta_k \leq 0$ and therefore $p \succeq \delta_k$. By Axiom 6, $p \succeq \delta_{k-1}$, and hence $\alpha_{\delta_{k-1}} \geq \alpha_p$. Since $\alpha_{\delta_{k-1}} \cdot \delta_{k-1} = 0$, $\alpha_{\delta_{k-1}}(k-1) = 0$, and therefore $\alpha_p(k-1) \leq 0$. Now suppose $\alpha_p(k) < 0$. By the preceding argument, $\alpha_p(k-1) \leq 0$. If $\alpha_p(k-1) = 0$, then $\alpha_p \cdot \delta_{k-1} = 0$ and so $p \sim \delta_{k-1}$. On the other hand, $\alpha_p \cdot \delta_k < 0$ and therefore $p \succ \delta_k$, which implies $\delta_{k-1} \succ \delta_k$ in violation of Axiom 6. Therefore, $\alpha_p(k-1) < 0$ and so α_p is single-crossing.

For the converse, suppose α_p is single-crossing for all p and consider indeces $n > m$. Since $\alpha_{\delta_n}\delta_n = 0$, $\alpha_{\delta_n}(n) = 0$. By single-crossing, $\alpha_{\delta_n}(m) \leq 0$, and therefore $\alpha_{\delta_n}\delta_m \leq 0$, which implies $\delta_n \succeq \delta_m$. Therefore, $\succeq$ satisfies Axiom 6. $\square$

Lemma 2 shows that indifference curves of a monotone betweenness preference can be represented by single-crossing vectors. We also require a converse, showing that any single-crossing vector can be interpreted as representing the indifference curve of a monotone betweenness preference. The following lemma provides one way to complete the map of indifference curves for a monotone betweenness preference.

Lemma 3. *If $\alpha \in \mathcal{C}(p^*)$ for some lottery p^*, then there is a monotone betweenness preference $\succeq$ such that $q \overset{\succsim}{\precsim} p^*$ if and only if $\alpha \cdot q \gtreqless 0$.*

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[5]By the equivalence $[q \overset{\succsim}{\precsim} p \;\Leftrightarrow\; \alpha_p \cdot q \gtreqless 0]$ we mean that (i) $q \succ p$ if and only if $\alpha_p \cdot q > 0$, (ii) $q \sim p$ if and only if $\alpha_p \cdot q = 0$, and (iii) $q \prec p$ if and only if $\alpha_p \cdot q < 0$. Similarly, by the equivalence $[q \overset{\succsim}{\precsim} p \;\Leftrightarrow\; \alpha_p \gtreqless \alpha_q]$ we mean that (i) $q \succ p$ if and only if $\alpha_p > \alpha_q$, (ii) $q \sim p$ if and only if $\alpha_p = \alpha_q$, and (iii) $q \prec p$ if and only if $\alpha_p < \alpha_q$, where $\alpha > \alpha'$ means $\alpha \geq \alpha'$ and $\alpha \neq \alpha'$.

[6]For a linear preference, there is a simple relation across the indifference curves: for any two lotteries p and q, α_p is a translation of α_q, which immediately implies that $\left(\{\alpha_p\}_{p\in\Delta}, \geq\right)$ is a chain. Without the independence axiom, it is not possible to represent the map of indifference curves by translation, but Lemma 4 in Siga and Mihm (2020) shows that the weak order and betweenness axioms still ensure that hyperplanes can be chosen so that $\left(\{\alpha_p\}_{p\in\Delta}, \geq\right)$ is a chain.

Proof. We start by using $\alpha \in \mathcal{C}(p^*)$ to construct a chain of single-crossing vectors, which we can then use to define a vector representation for a monotone betweenness preference. Start by defining the vector $\tilde{\alpha}$ by

$$\tilde{\alpha}(k) \equiv \begin{cases} -\alpha(k) & \text{if } \alpha(k) < 0 \\ 1 & \text{if } \alpha(k) = 0 \\ \alpha(k) & \text{if } \alpha(k) > 0 \end{cases}, \tag{8}$$

and observe the following for $k = 1, ..., K$ and $\lambda \in (0, 1]$:

(i) $\lambda\tilde{\alpha}(k) + (1-\lambda)\alpha(k)$ is strictly increasing in λ if $\alpha(k) \leq 0$, and constant if $\alpha(k) > 0$;

(ii) $\lambda(-\tilde{\alpha}(k)) + (1-\lambda)\alpha(k)$ is strictly decreasing in λ if $\alpha(k) \geq 0$, and constant if $\alpha(k) < 0$;

(iii) if $\alpha(k) > 0$, then $\lambda\tilde{\alpha}(k) + (1-\lambda)\alpha(k) > 0$, and $\lambda(-\tilde{\alpha}(k)) + (1-\lambda)\alpha(k) \lesseqgtr 0 \Leftrightarrow \lambda \gtreqless 1/2$;

(iv) if $\alpha(k) = 0$, then $\lambda\tilde{\alpha}(k) + (1-\lambda)\alpha(k) > 0$, and $\lambda(-\tilde{\alpha}(k)) + (1-\lambda)\alpha(k) < 0$; and

(v) if $\alpha(k) < 0$, then $\lambda\tilde{\alpha}(k) + (1-\lambda)\alpha(k) \lesseqgtr 0 \Leftrightarrow \lambda \gtreqless 1/2$, and $\lambda(-\tilde{\alpha}(k)) + (1-\lambda)\alpha(k) < 0$.

Observations (iii)—(v) imply that, for all $\lambda \in (0, 1/2]$, the vectors $\lambda\tilde{\alpha} + (1-\lambda)\alpha$ and $\lambda(-\tilde{\alpha}) + (1-\lambda)\alpha$ are non-zero and single-crossing. They are also non-zero and single-crossing when $\lambda = 0$ because they are both equal to α. Observations (i) and (ii) imply that, for the standard vector order $\geq$, we obtain a chain (from largest to smallest) as we first vary λ from 1/2 to 0 for the vectors $\lambda\tilde{\alpha}(k) + (1-\lambda)\alpha(k)$, and then vary λ from 0 to 1/2 for vectors $\lambda(-\tilde{\alpha}(k)) + (1-\lambda)\alpha(k)$. We use this chain of non-zero single-crossing vectors to construct a vector-representation by assigning one vector to each lottery p.

- For lottery p such that $\alpha \cdot p = 0$, let $\alpha_p \equiv \alpha$.
- For lottery p' such that $\alpha \cdot p' > 0$, observations (i)–(v) imply that there exists a unique $\lambda_{p'} \in (0, 1/2]$ such that $\lambda_{p'}(-\tilde{\alpha}) \cdot p' + (1-\lambda_{p'})\alpha \cdot p' = 0$, and let $\alpha_{p'} \equiv \lambda_{p'}(-\tilde{\alpha}) + (1-\lambda_{p'})\alpha$.
- For lottery p'' such that $\alpha \cdot p'' < 0$, observations (i)–(v) imply that there exists a unique $\lambda_{p''} \in (0, 1/2]$ such that $\lambda_{p''}\tilde{\alpha} \cdot p'' + (1-\lambda_{p''})\alpha = 0$, and let $\alpha_{p''} \equiv \lambda_{p''}\tilde{\alpha} + (1-\lambda_{p''})\alpha$.

We thereby obtain a continuous mapping from Δ to $\mathbb{R}^K$ such that $\{\alpha_p\}_{p\in\Delta}$ is a chain of non-zero single-crossing vectors. The vector α_p can be interpreted as the norm of a hyperplane with constant 0, which represents the indifference curve for lottery p. Note that, in the second bullet point, $\lambda'_p = 1/2$ for any lottery p' with support concentrated on $\{k : \alpha(k) > 0\}$, and so all such lotteries are on the same indifference curve. Likewise, in the third bullet point, $\lambda_{p''} = 1/2$ for any lottery p'' with support concentrated on $\{k : \alpha(k) < 0\}$, and so all such lotteries are on the same indifference curve. To complete the map of indifference curves, we are pivoting at the origin, which is outside the simplex and lies on all three hyperplanes with norms α, $(1/2)\tilde{\alpha} + (1/2)\alpha$, and $(1/2)(-\tilde{\alpha}) + (1/2)\alpha$.

Finally, we use the vector-representation to define the binary relation $\succeq$ on Δ by $q \succeq p$ if and only if $\alpha_p \cdot q \geq 0$. By construction, (i) $q \succeq p$ if and only if $\alpha_p \geq \alpha_q$, (ii) the mapping $p \mapsto \alpha_p$ is continuous, (iii) $\alpha_{\delta_1} \cdot \delta_K > 0$, (iv) $p \succeq\!\!\!\!\!\precsim q$ implies $p \succeq\!\!\!\!\!\precsim \theta p + (1-\theta) q \succeq\!\!\!\!\!\precsim q$ for $\theta \in (0,1)$, and (v) α_p is single-crossing for all p. Property (i) implies that $\succeq$ satisfies Axiom 1, because $(\{\alpha_p\}_{p \in \Delta}, \geq)$ is a chain; property (ii) implies Axiom 2; property (iii) implies Axiom 3; property (iv) implies Axiom 5; and property (v) implies Axiom 6 (by Lemma 2). Therefore, $\succeq$ is a monotone betweenness preference such that $q \succeq\!\!\!\!\!\precsim p^*$ if and only if $\alpha \cdot q \gtreqless 0$. □

Proof of Theorem 1: The proof follows directly from Lemmas 1–3, and we argue the contrapositive for each direction. Suppose p does not likelihood-ratio dominate q. By Lemma 1, there is $\alpha \in \mathcal{C}(q)$ such that $\alpha \cdot p < 0$. Hence, by Lemma 3, we can construct a monotone betweenness preference $\succeq$ such that $q \succ p$. For the converse, suppose there is a monotone betweenness preference $\succeq$ such that $q \succ p$. Using the vector-representation in Lemma 2, $\alpha_q \cdot p < 0$ and $\alpha_q \in \mathcal{C}(q)$. Hence, by Lemma 1, p does not likelihood-ratio dominate q. □

Discussion: Theorem 1 establishes a direct connection between the likelihood-ratio order and an economically meaningful class of preferences, which occupy a central role in the literature on decision making under risk.

On one hand, the characterization provides new insights on the likelihood-ratio order, which has many applications in mechanism design and information economics. For instance, consider a standard model of information, where signals provide information about an unknown outcome of interest. The information structure satisfies the *monotone likelihood-ratio property (MLRP)* if outcomes and signals are ordered such that $P(.|s)$ likelihood-ratio dominates $P(.|s')$ whenever $s \trianglerighteq s'$, where $P(.|.)$ is the conditional distribution over outcomes given a signal. Milgrom (1981) introduces the MLRP for problems in information economics and mechanism design to formalize the idea that signals convey more or less favorable news about outcomes, because higher signals induce uniformly more optimistic posteriors. Theorem 1 clarifies the behavioral meaning of the intuitive idea that signals convey more or less favorable news: information satisfies the MLRP if and only if any decision maker with monotone betweenness preferences prefers the conditional distribution over outcomes given signal s to the conditional distribution given signal s'. For economic applications, Theorem 1 therefore identifies the domain of preferences where the MLRP guarantees an unambiguous favorability ranking over signals. Unlike stochastic dominance, the favorability ranking extends to non-expected utility preferences; however, beyond the class of betweenness preferences, decision makers will disagree about the favorability of signals, and so the MLRP no longer reflects the idea that one signal conveys better news than another.

Understanding the behavioral implications of the likelihood-ratio order is important because, along with stochastic dominance, likelihood-ratio dominance is arguably the most prominent stochastic order in economic applications. As a corollary, Theorem 1 also has implications for other stochastic orders in probability and statistics, such as the hazard-rate and reverse hazard-rate order. Similar to stochastic dominance and likelihood-ratio dominance, the hazard-rate and reverse hazard-rate orders are defined by a system of linear inequalities with respect to a given order over outcomes (see, e.g., Shaked and Shanthikumar, 2007, Chapter 1), so that upper contour sets can be described by an intersection of half-spaces. Moreover, likelihood-ratio dominance implies both hazard-rate and reverse hazard-rate dominance, which in turn imply stochastic dominance. As a result, the hazard-rate and reverse hazard-rate orders must be characterized by a unanimity property for a class of monotone preferences that is larger than expected utility but smaller than the class of betweenness preferences.

On the other hand, Theorem 1 also provides a simple non-parametric prediction for behavior under risk. When more money is preferred to less, stochastic dominance provides a simple testable prediction of the expected utility hypothesis. For a binary choice between lotteries, monotonicity has implications only when both lotteries are degenerate, while expected utility has no implications for a binary choice problem. However, in combination, monotone expected utility implies that p must be chosen from $\{p, q\}$ whenever p stochastically dominates q, offering a falsifiable prediction. Betweenness preferences provide a tractable remedy for some of the descriptive failures of expected utility, and Theorem 1 identifies the additional admissible behaviors: choices that are inconsistent with stochastic dominance but respect likelihood-ratio dominance. In particular, if p likelihood-ratio dominates q, then not choosing p from $\{p, q\}$ is inconsistent with a monotone betweenness preference, and this prediction is the only testable implication of monotone betweenness preferences for a binary choice problem. Our characterization of the likelihood-ratio order therefore offers a simple testable prediction for an important class of non-expected utility preferences.

References

Allais, Maurice. 1953. "Le Comportement de l'Homme Rationnel Devant le Risque: Critique des Postulats et Axiomes de l'École Américaine." *Econometrica*, 21: 503–546.

Bikhchandani, Sushil, Uzi Segal, and Sunil Sharma. 1992. "Stochastic Dominance under Bayesian Learning." *Journal of Economic Theory*, 56: 352–377.

Cerreia-Vioglio, Simone, David Dillenberger, and Pietro Ortoleva. 2020. "An Explicit Representation for Disappointment Aversion and Other Betweenness Preferences." forthcoming *Theoretical Economics*.

Chan, Wai, Frank Proschan, and Jayaram Sethuraman. 1990. "Convex-ordering among Functions, with Applications to Reliability and Mathematical Statistics." *Lecture Notes-Monograph Series*, 16: 121–134.

Chateauneuf, Alain, Michele Cohen, and Isaac Meilijson. 2004. "Four Notions of Mean-preserving Increase in Risk, Risk Attitudes and Applications to the Rank-dependent Expected Utility Model." *Journal of Mathematical Economics*, 40(5): 547–571.

Chew, Soo Hong. 1983. "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox." *Econometrica*, 51: 1065–1092.

Dekel, Eddie. 1986. "An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom." *Journal of Economic Theory*, 40(2): 304 – 318.

Gul, Faruk. 1991. "A Theory of Disappointment Aversion." *Econometrica*, 59: 667–686.

Hadar, Josef, and William R. Russell. 1969. "Rules for Ordering Uncertain Prospects." *The American Economic Review*, 59(1): 25–34.

Kartik, Navin, SangMok Lee Lee, and Daniel Rappoport. 2019. "Single-Crossing Differences on Distributions." mimeo, Columbia University.

Lehrer, Ehud, and Tao Wang. 2020. "Strong Stochastic Dominance." mimeo.

Machina, Mark J. 1982. "Expected Utility Analysis without the Independence Axiom." *Econometrica*, 50: 277–323.

Milgrom, Paul R. 1981. "Good News and Bad News: Representation Theorems and Applications." *Bell Journal of Economics*, 12(2): 380–391.

Quah, John K-H, and Bruno Strulovici. 2012. "Aggregating the Single Crossing Property." *Econometrica*, 80(5): 2333–2348.

Quirk, James P., and Rubin Saposnik. 1962. "Admissibility and Measurable Utility Functions." *The Review of Economic Studies*, 29(2): 140–146.

Shaked, Moshe, and George Shanthikumar. 2007. *Stochastic Orders.* Springer Science & Business Media, New York.

Siga, Lucas, and Maximilian Mihm. 2020. "Information Aggregation in Competitive Markets." forthcoming *Theoretical Economics.*

Starmer, Chris. 2000. "Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk." *Journal of Economic Literature*, 38(2): 332–382.