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An Axiomatic Characterization of Bayesian Updating

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ABSTRACT

We provide an axiomatic characterization of Bayesian updating, viewed as a mapping from prior beliefs and new information to posteriors, which is disentangled from any reference to preferences. Bayesian updating is characterized by Non-Innovativeness (events considered impossible in the prior remain impossible in the posterior), Dropping (events contradicted by new evidence are considered impossible in the posterior), and Proportionality (for other events, the posterior simply rescales the prior's probabilities proportionally). The result clarifies the differences between the normative Bayesian benchmark, alternative models, and actual human behavior.

1. Introduction

When confronted with new evidence, a rational decision maker will update her beliefs. Normatively, belief updating is described by Bayes' rule, which pins down the correct balance between prior beliefs and new evidence. Yet, people are not Bayesian. Overwhelming evidence shows that decision makers routinely fall prey to a host of heuristics and biases (e.g., Edwards, 1968; Kahneman and Tversky, 1972; Grether, 1980; Camerer, 1987; Achtziger and Alós-Ferrer, 2014), which can explain a wide range of economic phenomena (e.g. Shleifer, 2000; Fama, 1998; Barberis et al., 1998). To capture and forecast behavior, descriptive models postulate specific deviations from Bayesian updating reflecting received evidence (e.g., Epstein, 2006; Bordalo et al., 2016; Gennaioli and Shleifer, 2010).

When a normative model fails to appropriately describe human behavior, axiomatic characterizations clarify which elements of the model are violated in different circumstances. Only a full characterization allows us to understand which are the separating properties when an alternative model is proposed. Previous work has identified joint characterizations of Bayesian updating *and* preferences on a choice domain. Most notably, Ghirardato (2002) provides a characterization of subjective expected utility and Bayesian updating in conditional decision problems, i.e. preferences over acts (Savage, 1954) conditional on the revelation of an event. However, many empirical demonstrations of non-Bayesian behavior entail tasks defined purely on the domain of probability judgments, or where payment involves dominated choices, as estimating which of two options is more probable (e.g., Kahneman and Tversky, 1972; Grether, 1980, 1992). For this reason, it is valuable to consider Bayesian updating without reference to choices or preferences.

The objective of this manuscript is to present a simple axiomatic characterization of Bayesian updating of beliefs on an abstract space. Our approach separates the issue of belief updating entirely from any reference to choice or preference, which necessarily entail a full specification of the choice domain. Instead, Bayesian updating is conceived simply as a particular case of an updating function, which maps priors and observation ("tests") to posteriors. In this abstract framework, we show that Bayesian updating is fully characterized by three natural properties. First, events which are considered impossible in the prior cannot be assigned positive probability in the posterior (Non-Innovativeness, NINN). Second, events which are contradicted by new evidence are considered impossible in the posterior (Dropping, DROP). Third, for all other events, updating is simply a proportional rescaling of the prior (Proportionality, PROP). The character of Bayes' rule as exactly balancing the prior and the new evidence clearly lies with this latter axiom but, as we show, all three axioms are necessary and independent of each other.

The axioms are, of course, not surprising. Conceptually, they will be familiar to specialists as they are rooted in classical decision-theoretic work and can be traced back and compared to insights going back as far as de Finetti (1937). However, to the best of our knowledge, our paper is the first to formulate a characterization within a probability-updating framework disentangled from any preference representation. It is immediate that each of the axioms is an implication of Bayesian updating. The contribution is to show that these axioms are the *only* implications of Bayesian updating: within an abstract framework, any updating function that satisfies these axioms must be Bayesian,

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and so violations of Bayesian updating can be traced back exactly to violations of one of these axioms.

Of course, connections to joint characterizations of updating and preferences exist. Appendix C shows that, if a conditional preference system over acts fulfills a weakening of the Dynamic Consistency axiom in Ghirardato (2002), every representation through (conditional) subjective expected utility must involve beliefs which fulfill our axioms, hence are generated by Bayesian updating.

In Section 3, we make explicit that the characterization assumes that updating respects the additivity of the posteriors. Adding this condition as an axiom in a more general setting results again in a characterization, which is of interest for empirical applications where additivity might be violated. In Section 4, we illustrate how some influential empirical analyses and at least one prominent model of non-Bayesian updating essentially postulate violations of proportionality (at least as long as additivity is not a concern). The Appendix contains the proofs, examples showing the independence of the axioms, and the relation to dynamic consistency.

2. Axioms and characterization

Let $(\Omega, \mathcal{C})$ be a measurable space. We assume that the space is finitely generated (but not necessarily finite[1]). This means that there is a finite set of *elementary events* (or *categories*) $\mathcal{A}$ which partition Ω and generate the algebra $\mathcal{C}$, e.g., the possible events in an experiment. That is, $\mathcal{A} \subseteq \mathcal{C}$, $A \cap A' = \emptyset$ for all $A, A' \in \mathcal{A}$ with $A \neq A'$, and every event $C \in \mathcal{C}$ is equal to the union of the elementary events it contains, $C = \bigcup_{A \in \mathcal{A}(C)} A$ where $\mathcal{A}(C) = \{A \in \mathcal{A} \mid A \subseteq C\}$.

A *test* (or criterion) is any nonempty event $E \in \mathcal{C}$. As is standard, we identify the set E with the event that the test is positive, i.e. E occurs. A *belief* is any probability measure π on $\mathcal{C}$. Denote the set of beliefs over categories by Π. Since any event E is a union of (disjoint) elementary events, we obtain that $\pi(E) = \sum_{A \in \mathcal{A}(E)} \pi(A)$.

Let $\mathcal{D} = \{(\pi, E) \in \Pi \times \mathcal{C} \mid \pi(E) > 0\}$ denote the subset of $\Pi \times \mathcal{C}$ with non-counterfactual tests. An *updating function* is a mapping

$$f : \mathcal{D} \mapsto \Pi$$

with the interpretation that $f(\pi, E)(C)$ is the posterior belief that an event C holds given that the prior was π and the test E has been observed to occur. *Bayesian updating* is the updating function f_B given by

$$f_B(\pi, E)(C) = \frac{\pi(C \cap E)}{\pi(E)} \tag{1}$$

for any $C \in \mathcal{C}$, $\pi \in \Pi$ and $E \in \mathcal{C}$ such that $\pi(E) > 0$.

Consider the following axiomatic properties. *Non-Innovative ness*, states that if a category has prior probability 0, it also has posterior probability 0 after any observed test. *Dropping*, states that when some event E is observed, any category not fulfilling E must have posterior probability 0. Finally, *Proportionality*, states that, given any two categories that both fulfill E and that have positive probability in the prior, the ratio of posterior probabilities after observing E is equal to the ratio of prior probabilities.[2]

Axiom (*NINN*). *Non-innovativeness:* For all $(\pi, E) \in \mathcal{D}$ and $A \in \mathcal{A}$,

$$\pi(A) = 0 \quad \Rightarrow \quad f(\pi, E)(A) = 0.$$

Axiom (*DROP*). *Dropping:* For all $(\pi, E) \in \mathcal{D}$, and $A \in \mathcal{A}$,

$$A \cap E = \emptyset \quad \Rightarrow \quad f(\pi, E)(A) = 0.$$

Axiom (*PROP*). *Proportionality:* For all $(\pi, E) \in \mathcal{D}$ and $A, A' \in \mathcal{A}$ with $A, A' \subseteq E$,

$$\pi(A) > 0, \pi(A') > 0 \text{ and } f(\pi, E)(A) > 0 \ \Rightarrow \ \frac{f(\pi, E)(A')}{f(\pi, E)(A)} = \frac{\pi(A')}{\pi(A)}.$$

Bayesian updating is the unique updating function that fulfills these three axioms.

Theorem 1. *An updating function f is Bayesian ($f = f_B$) if and only if it satisfies NINN, DROP, and PROP.*

Remark 1. The proof of Theorem 1 (see Appendix) also applies if a specific prior π (or subset of priors) is fixed, i.e., for an updating function defined on a restricted domain that does not allow for all possible priors.

Although the setting is quite different, it is possible to make a connection between PROP and Luce's Independence of Irrelevant Alternatives (IIA) axiom from the stochastic choice literature. Let $\mathcal{B}$ be a finite set of alternatives, $\mathcal{M}$ be the set of all menus (sets of alternatives), and let $P(a, M)$ denote the probability that a decision maker chooses alternative $a \in \mathcal{B}$ from menu $M \in \mathcal{M}$. The IIA axiom states that if $a, a' \in M \cap M'$, then

$$\frac{P(a, M)}{P(a', M)} = \frac{P(a, M')}{P(a', M')}.$$

To illustrate the connection, we can identify each $a \in B$ with an elementary event of the form "the decision maker chooses a", and each menu $M \in \mathcal{M}$ with the test "a menu $M \in \mathcal{M}$ has been offered". Let f be an updating function representing how an observer updates beliefs on the decision maker's choice after being informed of the menu. Then, $\pi(a) = P(a, \mathcal{B})$ and $f(\pi, M)(a) = P(a, M)$ for $a \in \mathcal{B}$, $M \in \mathcal{M}$. PROP hence states that

$$\frac{P(a, M)}{P(a', M)} = \frac{P(a, \mathcal{B})}{P(a', \mathcal{B})}$$

for any menu M with $a \in A$, and IIA follows.

3. Additivity of the posteriors

In the previous sections, we conceptualized updating functions as mappings within the set of probability distributions. This entails the implicit assumption that posterior beliefs are additive. In practice, decision makers could violate this implicit assumption. For instance, following Ellsberg (1961) and Schmeidler (1989), a large literature on ambiguity aversion has considered the economic implications of nonadditive beliefs in financial decision making, market exchange, and other fields (see, e.g., Dow and d.C. Werlang, 1992; Rigotti and Shannon, 2005; Easley and O'Hara, 2009; Mihm, 2016). It is hence worth making the additivity property explicit in our setting.

[1] For instance, the underlying set of events might be infinite, but only a finite number of events are observable. This is typical in experiments and problems where the observer wants to learn the preferences of a decision maker. As another example, multiple price lists to measure risk preferences or other preference parameters (Holt and Laury, 2002) provide only a finite partition of the preference space.

[2] PROP and NINN could be combined as a single axiom by dropping the requirement that $\pi(A') > 0$ from the statement of PROP. We keep these separate since NINN is an intuitive condition that is unlikely to be violated empirically, while PROP captures the character of Bayes' rule, which decision makers often violate in practice.

Given a finitely-generated measurable space $(\Omega, \mathcal{C})$ as in Section 2, let $\hat{\Pi}$ be the set of (potentially) non-additive beliefs, i.e. the set of functions $\hat{\pi} : \mathcal{C} \mapsto [0, 1]$ such that $\hat{\pi}(\emptyset) = 0$ and $\hat{\pi}(\Omega) = 1$. A *nonadditive updating function* is a mapping

$$\hat{f} : \mathcal{D} \mapsto \hat{\Pi}$$

where $\mathcal{D}$ is as in Section 2. For nonadditive updating functions, additivity is then an additional implication of Bayesian updating.[3]

Axiom (*Additivity*). For every $(\pi, E) \in \mathcal{D}$ and every collection of disjoint events $C_1, \ldots, C_n \in \mathcal{C}$, $f(\pi, E)\left(\bigcup_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} f(\pi, E)(C_i)$.

The following is an immediate corollary of Theorem 1.

Corollary 1. *A nonadditive updating function $\hat{f}$ is Bayesian ($\hat{f} = f_B$) if and only if it satisfies Additivity, NINN, DROP, and PROP.*

This corollary is, of course, a straightforward observation. The value of making it explicit lies in allowing for violations of additivity as a potential explanation of empirical failures of Bayesian updating (see, e.g., Example 2 below). For instance, it seems plausible – and may be interesting to explore empirically – that a decision maker could maintain proportionality when they update their beliefs for elementary events (hence, satisfying PROP) but be unable to update their beliefs properly for compound events while maintaining the implications of proportionality (hence, violating Additivity). The following example provides an illustration. It also shows how nonadditive beliefs could emerge from updating, even for a decision maker who has an additive prior belief and has an updating function that satisfies the key axioms of Bayesian updating (NINN, DROP and PROP) identified in Section 2.

Example 1. Let $n = |\mathcal{A}|$ be the number of categories, and let $\alpha : \{1, \ldots, n\} \mapsto [0, 1]$ be any decreasing function with $\alpha(1) = 1$ and $\alpha(n) = 0$. Consider the nonadditive updating function given by $f(\pi, E)(\emptyset) = 0$ and

$$f(\pi, E)(C) = \alpha(|\mathcal{A}(C)|) f_B(\pi, C) + (1 - \alpha(|\mathcal{A}(C)|))\pi(C)$$

for any $(\pi, E) \in \mathcal{D}$ and nonempty $C \in \mathcal{C}$. A decision maker with this nonadditive updating function correctly updates the probability of categories, but has increasing difficulties with more complicated events (those containing multiple categories). For compound events, the decision maker is anchored on the prior probabilities and fails to update fully. This has the consequence that additivity is violated by the posterior beliefs, even though NINN, DROP, and PROP are satisfied.

4. Relation to the empirical literature

The following example illustrates the connection to the empirical literature.

Example 2. Grether (1980, 1992) carried out influential experiments on deviations from Bayesian updating arising from the representativeness heuristic (Kahneman and Tversky, 1972) and conservatism (Edwards, 1968). In the experiments, participants had to guess from which of two urns (each containing black and white balls in certain proportions) a given sample of balls had been extracted (with replacement). Let A denote the event that the first urn is the actually used one, and $\Omega \setminus A$ the event that the actual urn is the second one. Denote by E the event corresponding to the sample that has actually been extracted. Participants were informed of the probability that the first urn was used, i.e. a prior $(\pi(A), \pi(\Omega \setminus A))$ and of the composition of the urns, hence of $\pi(A \cap E), \pi((\Omega \setminus A) \cap E)$.

Grether (1980) postulates that the actual, subjective posterior odds for A are given by

$$\frac{\hat{\pi}(A)}{\hat{\pi}(\Omega \setminus A)} = \alpha \cdot \left(\frac{\pi(E|A)}{\pi(E|\Omega \setminus A)}\right)^{\beta_1} \cdot \left(\frac{\pi(A)}{\pi(\Omega \setminus A)}\right)^{\beta_2} \tag{2}$$

plus observational error (the notation is adapted to ours here). Taking logarithms, this equation gives rise to a simple linear model which can be analyzed through probit or logit methods (since experiment participants only indicated which urn was more likely, the subjective posterior odds are a latent variable). Bayesian updating corresponds to $\alpha = 1$ and $\beta_1 = \beta_2 > 0$ (since the posterior is not observed), while representativeness implies $\beta_1 > \beta_2 \geq 0$, i.e. the sample information is overweighted. Analogously, conservatism (overweighting the prior) would correspond to $\beta_2 > \beta_1 \geq 0$.

Note that the likelihood ratio for A given E can be expressed as

$$\frac{\pi(E|A)}{\pi(E|\Omega \setminus A)} = \frac{\pi(A \cap E)}{\pi((\Omega \setminus A) \cap E)} \cdot \frac{\pi(\Omega \setminus A)}{\pi(A)}.$$

Thus, in our terms, and as long as additivity is assumed, the model defines an updating function such that

$$\frac{f(\pi, E)(A)}{1 - f(\pi, E)(A)} = \alpha \left(\frac{\pi(A)}{1 - \pi(A)}\right)^{\beta_2 - \beta_1} \left(\frac{\pi(A \cap E)}{\pi(E) - \pi(A \cap E)}\right)^{\beta_1}$$

or, equivalently, $f(\pi, E)(A)$ is obtained by evaluating the logistic function given by $L(x) = 1/(1 + (1/x))$ for $x > 0$ and $L(0) = 0$ on the right-hand-side of this last equation. For the empirically-relevant case with $\beta_1, \beta_2 > 0$ but $\beta_1 \neq \beta_2$, this updating function trivially fulfills NINN and DROP, but it fails PROP. In other words, the analysis of Grether (1980, 1992) implicitly assumed that empirical deviations from Bayesian updating would amount to violations of the PROP axiom. Axiom DROP could only be violated in the extreme case with $\beta_2 > \beta_1 = 0$, where the updating function would correspond to a "prior only" bias.

Remark 2. Experiments such as the ones discussed in Example 2 were designed under the implicit assumption that additivity would hold and are not well-suited to identify violations of additivity (this was not the experiments' objective). First, Grether (1980) only elicited whether an event A or its complement are more likely, i.e. whether $\hat{\pi}(A) > \hat{\pi}(\Omega \setminus A)$ or the opposite. Even if posterior beliefs are elicited, in a binary setting the typical approach is to elicit $\hat{\pi}(A)$ and assume $\hat{\pi}(\Omega \setminus A) = 1 - \hat{\pi}(A)$ in the analysis as, in a binary setting, it is reasonable to expect that experimental participants would naturally report additive posteriors for consistency reasons.

In principle, an experiment may find an apparent violation of proportionality which actually reflects a violation of additivity. This is because the events whose posterior probabilities are elicited are not necessarily categories (simplest events). In Example 2, categories would specify both the used urn and the color of the extracted balls, and the events used in the experiment (whether an urn has been used) are not categories. To make this point more clear, suppose that the decision maker follows the nonadditive updating rule in Example 1 and an experiment uses an event A which is actually a category, but $\Omega \setminus A$ contains $n - 1 \neq 1$ categories. Even though the updating function fulfills PROP, Eq. (2) will reveal a bias. As shown by Corollary 1 (in the presence of NINN and DROP), it is the combination of PROP and additivity that characterizes Bayesian updating.

[3] Note that finite additivity is enough since the measurable space is finitely generated.

In principle, it is possible to design experiments that disentangle PROP and additivity. Such experiments should involve elicited posteriors for more than two events, some of them being categories and some of them containing more than one category. The analogue of Eq. (2) for $\hat{\pi}(A)/\hat{\pi}(A')$, where both A and A' are categories, would yield a pure test of PROP, while violations of additivity (even if PROP holds) would be reflected in the corresponding equations where at least one of these events is not a category.

The following example singles out a prominent model of non-Bayesian updating and shows how it relates to our characterization.

Example 3. Gennaioli and Shleifer (2010) consider an agent who updates in the following way. There is a collection of given events or *hypotheses* $H_1, \ldots, H_R$. For simplicity, we assume they form a partition of $\mathcal{C}$. There is also a different collection of events of *scenarios* $S_1, \ldots, S_K$, which form an alternative partition of $\mathcal{C}$. Given a hypothesis H_r, there is a *representative scenario* $S(H_r, E)$ for each test E, which is the most likely scenario given the test:

$$S(H_r, E) = \arg\max_{S_1,\ldots,S_K} \frac{\pi(H_r \cap E \cap S_k)}{\pi(H_r \cap E \cap S_k) + \pi((\Omega \setminus H_r) \cap E \cap S_k)}$$

Given a test E, the agent represents the hypothesis H_r in terms $H_r^*(E) = H_r \cap E \cap S(H_r, E)$. The *representative state space* is $\tilde{\Omega}(E) = \bigcup_{r=1,\ldots,R} H_r^*(E)$. The agent then computes the probability of hypothesis H_r given test E as $\pi(H_r^*(E))/\pi(\tilde{\Omega}(E))$.

Consider the updating function defined by

$$f_{GS}(\pi, E)(A) = \begin{cases} \frac{\pi(A)}{\pi(\tilde{\Omega}(E))} & \text{if } A \subseteq \tilde{\Omega}(E) \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in \mathcal{A}$, and

$$f_{GS}(\pi, E)(C) = \sum \{ f_{GS}(\pi, E)(A) \mid A \in \mathcal{A}, A \subseteq C \}.$$

For a hypothesis H_r,

$$f_{GS}(\pi, E)(H_r) = \frac{\pi(H_r^*(E))}{\pi(\tilde{\Omega}(E))},$$

and so the updating function f_{GS} describes how the agent evaluates a hypothesis in the model Gennaioli and Shleifer (2010). This updating function satisfies NINN and DROP, but it does not, in general, satisfy PROP. For instance, if $\pi(A'), \pi(A) > 0$ for some $A', A \subset E$, $A' \notin \tilde{\Omega}(E)$ and $A \in \tilde{\Omega}(E)$, then

$$\frac{f_{GS}(\pi, E)(A')}{f_{GS}(\pi, E)(A)} = 0 \neq \frac{\pi(A')}{\pi(A)}.$$

5. Conclusion

Given the importance of Bayesian updating for economic analysis, and the widespread evidence that humans decision makers routinely depart from this normative ideal, a characterization of the updating function in itself (that is, independently of the representation of preferences) is interesting and potentially useful, e.g. for studies on probability judgments. Some previous works have asked whether a set of choices can be understood as utility maximization under Bayesian updating but imperfect perception (Caplin and Martin, 2015), or whether certain preferences over menus of lotteries can be represented as the result of maximization given non-Bayesian updating (Epstein, 2006). Here we ask a simpler question, and characterize Bayesian updating in the space of updating functions, separately from any reference to choices. The characterization makes evident that Bayesian updating reduces to a proportional rescaling of previously-held beliefs, once one discards events contradicted by the evidence or previously discarded, plus additivity of the posterior beliefs. While these properties are of course not surprising, the result makes clear that the characterization is separate and distinct from any result on preference representation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Main proof

Proof of Theorem 1. Bayesian updating, f_B, obviously satisfies NINN, DROP, and PROP. For the converse, let f be an updating function that satisfies those three properties. Let $\pi \in \Pi$ denote a prior and consider an arbitrary test $E \in \mathcal{C}$ with $\pi(E) > 0$. We need to show that $f(\pi, E)$ coincides with $f_B(\pi, E)$. The proof proceeds in five steps.

Step 1. We claim that there exists some category $A^* \in \mathcal{A}$ such that $A^* \subseteq E$, $\pi(A) > 0$, and $f(\pi, E)(A) > 0$. To see this, note that

$$1 = \sum_{A \in \mathcal{A}} f(\pi, E)(A) = \sum_{\{A \in \mathcal{A} \mid A \cap E = \emptyset\}} f(\pi, E)(A) + \sum_{A \in \mathcal{A}(E)} f(\pi, E)(A).$$

The first term on the right-hand side is equal to zero by DROP, hence, the second term on the right-hand side is equal to one, and in particular strictly positive. Thus, one of the summands must be strictly positive. That is, there exists $A^* \in \mathcal{A}$ such that $A^* \subseteq E$ and $f(\pi, E)(A^*) > 0$. By NINN, the latter implies $\pi(A^*) > 0$.

Step 2. For every $A \in \mathcal{A}$ $f(\pi, E)(A) = 0$ holds if and only if either $\pi(A) = 0$ or $A \cap E = \emptyset$.

If $\pi(A) = 0$ or $A \cap E = \emptyset$, it follows from NINN and DROP, respectively, that $f(\pi, E)(A) = 0$. To see the converse implication, suppose $f(\pi, E)(A) = 0$ but $A \subseteq E$. If $\pi(A) > 0$, then by PROP,

$$\frac{f(\pi, E)(A)}{f(\pi, E)(A^*)} = \frac{\pi(A)}{\pi(A^*)}$$

and hence $f(\pi, E)(A) > 0$, a contradiction which proves that $\pi(A) = 0$.

Step 3. If $A \in \mathcal{A}$ with $f(\pi, E)(A) = 0$, then $f_B(\pi, E)(A) = 0$.

This follows directly from Step 2 and the definition of f_B.

Step 4. If $A \in \mathcal{A}$ with $f(\pi, E)(A) > 0$, then $f(\pi, E)(A) = f_B(\pi, E)(A)$.

To see this, note that by Step 2 $\pi(A) > 0$ and $A \subseteq E$. By PROP, for each $A' \in \mathcal{A}$ with $A' \subseteq E$ and $\pi(A') > 0$, we have that

$$f(\pi, E)(A') = \frac{\pi(A')}{\pi(A)} f(\pi, E)(A)$$

which also holds if $\pi(A') = 0$ by NINN. Adding up over all categories $A' \in \mathcal{A}$ such that $A' \subseteq E$ yields

$$\sum_{A' \in \mathcal{A}(E)} f(\pi, E)(A') = \frac{f(\pi, E)(A)}{\pi(A)} \sum_{A' \in \mathcal{A}(E)} \pi(A') = \frac{f(\pi, E)(A)}{\pi(A)} \pi(E).$$

By DROP, the left-hand side is equal to 1, and so re-arranging terms yields

$$f(\pi, E)(A) = \frac{\pi(A)}{\pi(E)} = f_B(\pi, E)(A)$$

where the second equality follows from (1) for the particular case of elementary events.

Step 5. $f(\pi, E) = f_B(\pi, E)$.

It remains to show that $f(\pi, E)$ coincides with Bayesian updating as given in (1) for any arbitrary event $C \in \mathcal{C}$. Since $f(\pi, E)$ is a probability measure and $\mathcal{A}$ generates $\mathcal{C}$,

$$f(\pi, E)(C) = \sum_{A \in \mathcal{A}(C)} f(\pi, E)(A) = \sum_{A \in \mathcal{A}(C \cap E)} f(\pi, E)(A) + \sum \{f(\pi, E)(A) \mid A \in \mathcal{A}(C), A \cap E = \emptyset\}.$$

Since $f(\pi, E)(A) = f_B(\pi, E)(A)$ for all $A \in \mathcal{A}$ by Steps 3 and 4, and in particular since $f(\pi, E)(A) = 0$ whenever $A \in \mathcal{A}, A \cap E = \emptyset$, it follows that

$$f(\pi, E)(C) = \sum_{A \in \mathcal{A}(C \cap E)} f_B(\pi, E)(A) = \frac{1}{\pi(E)} \sum_{A \in \mathcal{A}(C \cap E)} \pi(A) = \frac{\pi(C \cap E)}{\pi(E)}$$

where the second equality follows from (1) for elementary events and the last equality follows because $C \cap E \in \mathcal{C}$ and the latter is generated by $\mathcal{A}$. We conclude that $f(\pi, E) = f_B(\pi, E)$ for all nonempty $E \in \mathcal{C}$, and the proof is complete. □

Appendix B. Independence of the axioms

The three following examples show that axioms NINN, DROP, and PROP are independent of each other, by exhibiting updating functions which fulfill two of those axioms but not the third. For the examples, let $n = |\mathcal{A}|$ be the number of elementary events.

Example 4. DROP and PROP do not imply NINN. Fix $\varepsilon > 0$ such that $\varepsilon < 1/n$. Let $\pi \in \Pi$. To construct the updating function, for each $E \in \mathcal{C}$ with $\pi(E) > 0$, let $N(\pi, E) = |\{A \in \mathcal{A} \mid \pi(A) = 0 \text{ and } A \subseteq E\}|$ and define, for each $A \in \mathcal{A}$,

$$f(\pi, E)(A) = \begin{cases} 0 & \text{if } A \cap E = \emptyset \\ (1 - \varepsilon \cdot N(\pi, E)) \frac{\pi(A)}{\pi(E)} & \text{if } A \subseteq E \text{ and } \pi(A) > 0 \\ \varepsilon & \text{if } A \subseteq E \text{ and } \pi(A) = 0. \end{cases}$$

which clearly yields a well-defined probability distribution. This rule sets the probability of all categories previously held as not possible but compatible with the new evidence to $\varepsilon > 0$, bringing them back as feasible hypotheses. It then updates the probability of all other categories according to a rescaling of Bayes' rule. Hence, PROP and DROP are satisfied, but NINN is not.

Example 5. PROP and NINN do not imply DROP. As in the previous example, fix $\varepsilon > 0$ such that $\varepsilon < 1/n$. For each $\pi \in \Pi$ and each $E \in \mathcal{C}$ such that $\pi(E) > 0$, let $M(\pi, E) = |\{A \in \mathcal{A} \mid \pi(A) > 0 \text{ and } A \cap E = \emptyset\}|$ and define, for each $A \in \mathcal{A}$,

$$f(\pi, E)(A) = \begin{cases} 0 & \text{if } \pi(A) = 0 \\ (1 - \varepsilon \cdot M(\pi, E)) \frac{\pi(A)}{\pi(E)} & \text{if } A \subseteq E \text{ and } \pi(A) > 0 \\ \varepsilon & \text{if } A \cap E = \emptyset \text{ and } \pi(A) > 0 \end{cases}$$

which again yields a well-defined probability distribution. This rule sets the probability of all categories previously held as possible but incompatible with the new evidence to $\varepsilon > 0$, and updates the probability of all other options according to a rescaling of Bayes' rule. Hence, PROP and NINN are satisfied, but DROP is not.

Example 6. DROP and NINN do not imply PROP. For each $\pi \in \Pi$ and each $E \in \mathcal{C}$ such that $\pi(E) > 0$, let $L(\pi, E) = |\{A \in \mathcal{A} \mid \pi(A) > 0 \text{ and } A \subseteq E\}|$ (which is a strictly positive integer since $\pi(E) > 0$) and define, for each $A \in \mathcal{A}$,

$$f(\pi, E)(A) = \begin{cases} 0 & \text{if } A \cap E = \emptyset \text{ or } \pi(A) = 0 \\ 1/L(\pi, E) & \text{if } A \subseteq E \text{ and } \pi(A) > 0 \end{cases}$$

which obviously yields a well-defined probability distribution. This rule drops all categories previously held impossible or incompatible with the new evidence, and prescribes a uniform distribution over the rest. Hence, it fulfills DROP and NINN by construction. However, if fails PROP. Suppose $\mathcal{A} = \{A_1, A_2, A_3\}$ and let π be given by $\pi(A_1) = 1/2$, $\pi(A_2) = 1/3$, $\pi(A_3) = 1/6$. Let $E = A_1 \cup A_2$. Then, $f(\pi, E)(A_1) = f(\pi, E)(A_2) = 1/2$, but PROP would imply $f(\pi, E)(A_1) = (3/2)f(\pi, E)(A_2)$.

The examples above are updating functions, i.e. they fulfill additivity. Example 1 in the main text shows that, for nonadditive updating functions, additivity is independent of the rest of the axioms.

Appendix C. Belief updating and preferences

For preferences over contingent claims (acts), there are well-known characterizations of subjective expected utility (Savage, 1954; Anscombe and Aumann, 1963) that imply Bayesian updating as a side-product, once one considers decisions conditional on the revelation of an event. Ghirardato (2002) observes that a joint characterization of subjective expected utility and Bayesian updating does not require the full force of the Savage axioms (Savage, 1954). Specifically, Ghirardato (2002) weakens the Savage axioms while adding two conditions. The first, called *Consequentialism*, states that the preference conditional on the revelation of a non-null event depends only on the implications of acts for that event. The second is a *Dynamic Consistency* condition stating that preferences after an event is revealed coincide with the preferences before the event is revealed if the consequences when the event does not happen are identical across the alternatives. This is related to other axioms in the literature, e.g. Myerson (1979) (substitution). Even though those axioms are formulated in terms of preferences, the second one can be shown to be conceptually related to our axioms. In particular, we now show that a weaker version of dynamic consistency implies all three of our axioms and, in this sense, Dynamic Consistency is stronger than needed to obtain Bayesian updating.

To formalize this statement, we briefly extend our setting to include preferences over acts. Let Z be an arbitrary *prize space*, and let an *act* be any mapping $x : \mathcal{A} \mapsto Z$, with the interpretation that the prize $x(A)$ obtains if the elementary event A occurs. Let $\mathcal{X}$ denote the set of acts. A *conditional preference system* over acts is a collection of preferences $\{\succeq_E \mid E \in \mathcal{C}\}$ over acts. It is called *nontrivial* if there exist at least two acts x, y such that $x \succ_\Omega y$. The (unconditional) preference $\succeq_\Omega$ is denoted simply by $\succeq$. A conditional preference system is said to have a *conditional subjective expected utility (C-SEU) representation* if there exists a probability measure π on $\mathcal{C}$, an updating function[4] f, and a utility function $u : Z \mapsto \mathbb{R}$ such that, for any event E with $\pi(E) > 0$ and any $x, y \in \mathcal{X}$,

$$x \succeq_E y \iff \sum_{A \in \mathcal{A}(E)} f(\pi, E)(A)u(x(A)) \geq \sum_{A \in \mathcal{A}(E)} f(\pi, E)(A)u(y(A)).$$

A C-SEU is *Bayesian* if $f = f_B$. Obviously, a C-SEU need not be Bayesian, since no discipline is imposed on the beliefs $f(\pi, E)$. Ghirardato (2002) characterizes Bayesian C-SEUs, simultaneously pinning down the (conditional) subjective expected utility form and the updating function. We now prove a related statement based on a weakening of Dynamic Consistency.

Given two acts $x, y \in \mathcal{X}$ and an event E, denote by xEy the act given by $xEy(A) = x(A)$ if $A \in \mathcal{A}(E)$ and $xEy(A) = y(A)$ if $A \notin \mathcal{A}(E)$.

[4] In this setting it is sufficient to specify the updating function for the given prior π (see Remark 1 in the main text).

Dynamic Consistency states that, for every (non-null[5]) event E and acts $x, y \in \mathcal{X}$,

$$x \succeq_E y \iff xEy \succeq y.$$

Hence, an act x is preferred to another act y given that an event E is known, if and only if the act giving the same consequences as x if E occurs and as y otherwise is also preferred to y.

We consider the following, weaker condition, stated in terms of the indifference relations $\sim_E$ derived from $\succeq_E$.

Axiom (*Weak Consistency*). For every $E \in \mathcal{C}$ and every $x, y \in \mathcal{X}$,

$$xEy \sim y \Rightarrow x \sim_E y.$$

That is, Weak Consistency requires only that, whenever two acts which coincide outside of an event E are indifferent, they are also indifferent after the event E is revealed to occur. This is weaker than Dynamic Consistency in that only one implication is required, and only for indifference.

This condition will be enough to establish our axioms, with the exception of an additional, technical condition guaranteeing that there are enough prizes to construct indifferences. To state this additional condition, abuse notation to denote by $z \in \mathcal{X}$ the constant act assigning prize $z \in Z$ to any elementary event $A \in \mathcal{A}$.

Axiom (*Bet Continuity*). For any two events $E_1, E_2 \in \mathcal{C}$, there exist prizes $z_0, z_1, z_2 \in Z$ such that

$$z_1 E_1 z_0 \sim z_2 E_2 z_0$$

and neither $z_1 \sim z_0$ nor $z_2 \sim z_0$.

Bet Continuity requires that, for any two events, there are two prizes such that betting on one event and obtaining the first prize if it occurs is indifferent to betting on the other event and obtaining the second prize, for a certain third outcome which occurs when a bet fails. For a C-SEU representation this merely means that there are outcomes such that $u(z_1)\pi(E_1) = u(z_2)\pi(E_2)$ (normalizing $u(z_0) = 0$).

Theorem 2. *Suppose a nontrivial conditional preference system over acts fulfills Weak Consistency and Bet Continuity. Then, any conditional subjective expected utility representation (if one exists) must be Bayesian.*

In conclusion, once a set of axioms is brought to bear to obtain a conditional subjective expected utility representation, Weak Consistency essentially suffices to guarantee that the beliefs involved in the representation fulfill NINN, DROP, and PROP, and hence the updating function in the representation must be Bayesian. In contrast, Consequentialism as defined by Ghirardato (2002) is irrelevant for establishing Bayesian updating in Theorem 2. Consequentialism states that if two acts coincide on a non-null event, the decision maker must be indifferent among them if that event is revealed to occur. However, once a conditional subjective expected utility is obtained, this property is immediate in terms of the representation and has no further implications for updating probabilities.

Remark 3. There is also a converse for Theorem 2. If a conditional preference system has a C-SEU representation with Bayesian updating, then it satisfies Weak Consistency. This direction is immediate because, for a C-SEU, Bayesian updating implies Dynamic Consistency (hence, Weak Consistency) by Theorem 1 in Ghirardato (2002).

[5] Given a preference $\succeq$ over $\mathcal{X}$, an event E is *null* if, for every $x_1, x_2, y_1, y_2 \in \mathcal{X}$, $x_1(\Omega \setminus E)y_1 \succeq x_2(\Omega \setminus E)y_2$ if and only if $x_1 \succeq x_2$. To avoid complications orhtogonal to our purposes, we formulate our Weak Consistency Axiom below for all events.

Proof of Theorem 2. Suppose a C-SEU representation exists. Weak Consistency implies that, for each event E and each pair of acts $x, y \in \mathcal{X}$,

$$\sum_{A\in\mathcal{A}(E)} u(x(A))\pi(A) = \sum_{A\in\mathcal{A}(E)} u(y(A))\pi(A)$$
$$\Rightarrow \sum_{A\in\mathcal{A}} u(x(A))f(\pi, E)(A) = \sum_{A\in\mathcal{A}} u(y(A))f(\pi, E)(A). \quad (3)$$

By nontriviality, there must exist two prizes $\bar{z}, \underline{z}$ such that $u(\underline{z}) < u(\bar{z})$, and without loss of generality we can rescale u so that $u(\underline{z}) = 0$ and $u(\bar{z}) = 1$.

We first prove NINN. Consider $A \in \mathcal{A}$ with $\pi(A) = 0$, and an arbitrary event $E \in \mathcal{C}$. Construct the acts x, y given by $x(A) = \bar{z}$, $x(A') = \underline{z}$ for all $A' \neq A$, and $y(A') = \underline{z}$ for all $A' \in \mathcal{A}$. Since $\pi(A) = 0$, it follows that $\sum_{A'\in\mathcal{A}(E)} u(x(A'))\pi(A') = \sum_{A'\in\mathcal{A}(E)} u(y(A'))\pi(A')$ and, applying (3), we obtain

$$f(\pi, E)(A) = \sum_{A'\in\mathcal{A}} u(x(A'))f(\pi, E)(A') = \sum_{A'\in\mathcal{A}} u(y(A'))f(\pi, E)(A') = 0,$$

implying NINN.

We now turn to DROP. Let $A \in \mathcal{A}$ and $E \in \mathcal{C}$ be such that $A \cap E = \emptyset$. Consider the acts x, y given by $x(A') = \bar{z}$ for all $A' \in \mathcal{A}(A \cup E)$, $x(A') = \underline{z}$ for all $A' \notin \mathcal{A}(A \cup E)$, $y(A') = \bar{z}$ for all $A' \in \mathcal{A}(E)$, and $y(A') = \underline{z}$ for all $A' \notin \mathcal{A}(E)$. Then, trivially, $\sum_{A'\in\mathcal{A}(E)} u(x(A'))\pi(A') = \sum_{A'\in\mathcal{A}(E)} u(y(A'))\pi(A')$ and, applying (3), we obtain $f(\pi, E)(A) + f(\pi, E)(E) = f(\pi, E)(E)$, i.e. $f(\pi, E)(A) = 0$, implying DROP.

Finally, we prove PROP. Consider $A_1, A_2 \in \mathcal{A}$ and $E \in \mathcal{C}$ with $A_1, A_2 \subseteq E$ and $\pi(A_1), \pi(A_2), f(\pi, E)(A_1) > 0$. By Bet Continuity, there exist prizes z_0, z_1, z_2 such that

$$u(z_1)\pi(A_1) + u(z_0)\pi(\Omega \setminus A_1) = u(z_2)\pi(A_2) + u(z_0)\pi(\Omega \setminus A_2).$$

and hence

$$[u(z_1) - u(z_0)]\pi(A_1) = [u(z_2) - u(z_0)]\pi(A_2) + u(z_0),$$

implying that

$$\frac{\pi(A_1)}{\pi(A_2)} = \frac{u(z_2) - u(z_0)}{u(z_1) - u(z_0)},$$

where $u(z_1) - u(z_0) \neq 0$ since $z_1 \sim z_0$ does not hold.

Consider the acts x, y given by $x(A_1) = z_1$, $x(A) = z_0$ for all $A \neq A_1$, $y(A_2) = z_2$, and $y(A) = z_0$ for all $A \neq A_2$. Then

$$\sum_{A\in\mathcal{A}(E)} x(A)\pi(A) = u(z_2)\pi(A_1) + u(z_0)\pi(\Omega \setminus A_1) =$$
$$u(z_2)\pi(A_2) + u(z_0)\pi(\Omega \setminus A_2) = \sum_{A\in\mathcal{A}(E)} y(A)\pi(A)$$

and, applying (3), we obtain

$$u(z_1)f(\pi, E)(A_1) + u(z_0)f(\pi, E)(\Omega \setminus A_1) = \sum_{A\in\mathcal{A}} u(x(A))f(\pi, E)(A) =$$
$$\sum_{A\in\mathcal{A}} u(y(A))f(\pi, E)(A) = u(z_2)f(\pi, E)(A_2) + u(z_0)f(\pi, E)(\Omega \setminus A_2),$$

that is,

$$[u(z_1) - u(z_0)]f(\pi, E)(A_1) = [u(z_2) - u(z_0)]f(\pi, E)(A_2),$$

implying that

$$\frac{f(\pi, E)(A_1)}{f(\pi, E)(A_2)} = \frac{u(z_2) - u(z_0)}{u(z_1) - u(z_0)} = \frac{\pi(A_1)}{\pi(A_2)}.$$

which shows PROP. □

References

Achtziger, A., Alós-Ferrer, C., 2014. Fast or rational? A response-times study of Bayesian updating. Manage. Sci. 60, 923–938.

Anscombe, F.J., Aumann, R.J., 1963. A definition of subjective probability. Ann. Math. Stat. 34, 199–205.

Barberis, N., Shleifer, A., Vishny, R., 1998. A model of investor sentiment. J. Financ. Econ. 49, 307–343.

Bordalo, P., Coffman, K., Shleifer, A., 2016. Stereotypes. Q. J. Econ. 131, 1753–1794.

Camerer, C.F., 1987. Do biases in probability judgment matter in markets? Experimental evidence. Amer. Econ. Rev. 77, 981–997.

Caplin, A., Martin, D., 2015. A testable theory of imperfect perception. Econom. J. 125, 18–202.

de Finetti, B., 1937. La prévision: ses lois logiques, ses sources subjectives. Ann. L'Inst. Henri Poincaré 7, 1–68.

Dow, J., d.C. Werlang, S.R., 1992. Uncertainty aversion, risk aversion, and the optimal choice of portfolio. Econometrica 60, 197–204.

Easley, D., O'Hara, M., 2009. Ambiguity and nonparticipation: The role of regulation. Rev. Financ. Stud. 22, 1817–1843.

Edwards, W., 1968. Conservatism in human information processing. In: Kleinmuntz, B. (Ed.), Formal Representation of Human Judgment. Wiley, New York, NY, pp. 17–52.

Ellsberg, D., 1961. Risk, ambiguity and the Savage axioms. Q. J. Econ. 75, 643–669.

Epstein, L.G., 2006. An axiomatic model of non-Bayesian updating. Rev. Econom. Stud. 73, 413–436.

Fama, E.F., 1998. Market efficiency, long-term returns, and behavioral finance. J. Financial Econ. 49, 283–306.

Gennaioli, N., Shleifer, A., 2010. What comes to mind. Q. J. Econ. 125, 1399–1433.

Ghirardato, P., 2002. Revisiting Savage in a conditional world. Econom. Theory 20, 83–92.

Grether, D.M., 1980. Bayes rule as a descriptive model: The representativeness Heuristic. Q. J. Econ. 95, 537–557.

Grether, D.M., 1992. Testing Bayes rule and the representativeness Heuristic: Some experimental evidence. J. Econ. Behav. Organ. 17, 31–57.

Holt, C.A., Laury, S.K., 2002. Risk aversion and incentive effects. Amer. Econ. Rev. 92, 1644–1655.

Kahneman, D., Tversky, A., 1972. Subjective probability: A judgment of representativeness. Cogn. Psychol. 3, 430–454.

Mihm, M., 2016. Reference dependent ambiguity. J. Econom. Theory 163, 495–524.

Myerson, R.B., 1979. An axiomatic derivation of subjective probability, utility, and evaluation functions. Theory Decis. 11, 339–352.

Rigotti, L., Shannon, C., 2005. Uncertainty and risk in financial markets. Econometrica 73, 203–243.

Savage, L.J., 1954. The Foundations of Statistics. John Wiley & Sons, New York.

Schmeidler, D., 1989. Subjective probability and expected utility without additivity. Econometrica 57, 571–587.

Shleifer, A., 2000. Inefficient Markets: An Introduction to Behavioural Finance. OUP Oxford, Oxford, UK.